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A

SYSTEM

OF

POPULAR TRIGONOMETRY,

BOTH PLANE AND SPHERICAL;

WITH

POPULAR TREATISES ON LOGARITHMS, AND THE
APPLICATION OF ALGEBRA TO GEOMETRY.

CONTAINING
IN A FEW LESSONS
AS MUCH AS IS
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OF EVERY
ART AND SCIENCE
IN ITS
LEADING TRUTHS AND GENERAL PRINCIPLES.

BY GEORGE DARLEY, A. B.

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LOGARITHMS.

The hardest part of this Science is its *name.* After what has been said on *Powers* and *Progressions,* in our ALGEBRA, the reader will find little difficulty in understanding the nature and use of LOGARITHMS. So much as is necessary and sufficient, we propose to explain in the present Treatise.

It is manifest from inspection that in a series of terms in *geometric* progression, the indices of the multiplying ratio themselves are a series in *arithmetic* progression; thus, if the former series be

$$a, \ ar^1, \ ar^2, \ ar^3, \ ar^4, \ ar^5, \ ar^6, \ \&c.$$

inasmuch as a is equal to $a \times 1$, or $a \times r^0$ (by *Obs.* ART. 37, ALG.), the series of indices will be

$$0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ \&c.$$

which is an arithmetic series. This corresponds exactly with the other, or more properly speaking, with r, the multiplying ratio, of the other; the *powers* of which in the successive terms are indicated, or numbered, by the corresponding terms of the second series. Hence originates the word Logarithms; scil. from two Greek words (*arithmos* and *logos*), signifying the numeration of a ratio. The terms of the arithmetic series are called logarithms of the multiplying ratio in the geometric series; thus 0 is the logarithm of r^0, 1 the log. of r^1, 2 the log. of r^2, &c.

It is also apparent that the same arithmetic series will correspond to the geometric series

$$r^0, \ r^1, \ r^2, \ r^3, \ r^4, \ r^5, \ r^6, \ \&c.$$

or to *any* geometric series whatever, the indices of whose multiplying ratio increase in like manner. Moreover, if there be geometric terms interposed between the above, thus,

$$r^0, \ r^{\frac{1}{2}}, \ r^1, \ r^{\frac{3}{2}}, \ r^2, \ r^{\frac{5}{2}}, \ r^3, \ r^{\frac{7}{2}}, \ r^4, \ \&c.$$

the indices or logarithms,

$$0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2,\ \tfrac{5}{2},\ 3,\ \tfrac{7}{2},\ 4,\ \&c.$$

will be still in arithmetic progression. This very simple correspondence between two such series furnishes a most powerful instrument of computation, as the reader will acknowledge when described.

If we multiply any two terms of the geometric series, as $r^{\frac{3}{2}}$, r^2, we get (by ART. 36, AL.) $r^{\frac{3}{2}+2}$, or $r^{\frac{7}{2}}$. Hence, instead of performing the actual multiplication, we need *only add* the logarithms of $r^{\frac{3}{2}}$ and r^2, scil. $\frac{3}{2}$ and 2, which would give $\frac{7}{2}$; and this being the logarithm of the product required, points out that product $r^{\frac{7}{2}}$ without any further trouble. By this means the simpler operation of Addition is made to serve in place of Multiplication, as far at least as respects the terms of the above geometric series, or any such that can be devised.

Suppose then we devise a geometric series containing certain numbers which we are anxious to multiply very briefly, the foregoing principle that the log. of the product of two numbers is equal to the sum of their logs. may be successfully applied. Suppose, for example, we choose the series

$$2^0,\ 2^1,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6,\ 2^7,\ 2^8,\ 2^9,\ 2^{10},\ \&c.$$

then the arithmetic series

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

will be respectively the logarithms of the following numbers,

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, &c.

which are only the terms of the first series raised to their several powers. Consequently, to multiply any two of these numbers, 16 by 64, we have only to add their logarithms 4 and 6, which gives 10; but as the sum of the logs. is equal to the log. of the product of 16 and 64, therefore the number of which 10 is the log. must be the product required, viz. 2^{10} or 1024. Now if we had a *table* with the numbers, 1, 2, 4, 8, &c. in one column, and their logarithms, 0, 1, 2, 3, &c. opposite to them in another, thus,

Numbers.	Logarithms.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
&c.	&c.

in order to multiply any two of these numbers, we should merely have to look at their opposite logarithms; add these, and their sum would be the logarithm opposite their product. Thus to multiply 8 and 32; the table shows 3 and 5 for their logarithms, the sum of these is 8, and 8 stands opposite to 256, therefore this is the product sought.

But why not choose such a geometric series as will contain *all* numbers, and by registering their logarithms in a table possess the same expeditious mode of multiplication? This in fact *has* been done; and the reader who asks himself the preceding question has involuntarily discovered, to a great extent, the use of Logarithms, which we promised to impart. There is probably, however, one circumstance on which he has not yet reflected, namely, that there is no geometric series which *can* furnish us with all numbers *exactly*. For instance, that power of 2 which is equal to 3, lies between 2^1 and 2^2; it is in fact a *surd* which cannot be expressed in finite rational terms. The most convenient geometrical series for our purpose is perhaps that which has for its terms the several powers of 10; at least it is that preferred by mathematicians, which may recommend it to our favour. Computists have found that the terms of this series are pretty nearly equal to the digital numbers.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

are respectively

$$10^0, 10^{\frac{3}{10}}, 10^{\frac{4}{10}}, 10^{\frac{6}{10}}, 10^{\frac{7}{10}}, 10^{\frac{77}{100}}, 10^{\frac{84}{100}}, 10^{\frac{90}{100}}, .10^{\frac{95}{100}},$$
$$10^1, \&c.$$

or more accurately, in decimal powers,

$1 = 10^{0,000000}$
$2 = 10^{0,301030}$
$3 = 10^{0,477121}$
$4 = 10^{0,602060}$
$5 = 10^{0,698970}$
$6 = 10^{0,778151}$
$7 = 10^{0,845098}$
$8 = 10^{0,903090}$
$9 = 10^{0,954243}$
$10 = 10^{1,000000}$
&c. &c.

Let us register these digital numbers and their logarithms in a tabular form; we shall then have a specimen of the logarithmic tables in common use.

Numbers.	Logarithms.
1	0 000000
2	·301030
3	·477121
4	·602060
5	·698970
6	·778151
7	·845098
8	·903090
9	·954243
10	1·000000
&c.	&c.

This table, like the preceding, furnishes us with the means of multiplying two numbers by means of addition. For, if the product of 2 and 4 be required, we have only to add their logs.; *scil.* ,301030 and ,602060, which gives ,903090; but the number of which this is the log must be the product required, and by the table the number appears to be 8, which is accordingly the true answer.

This is enough to explain the nature of logarithms, though not much calculated to exhibit their utility, the numbers we have employed being so easily multipliable. But the student will perceive from what has been said, that if the above table were carried on to the higher classes of numbers, the difficulty of multiplication would increase enormously, while that of addition would remain nearly the same. Had we, for example, to multiply 3297648501 by 98496087, it would not be so easy to do this by the common operation as to consult a logarithmic table, if we had one, add the logs. of these numbers, and see what other number in the table corresponded to the sum; for that would be the product sought. But the utility of logarithms does not end here, as we proceed to show.

The origin of the science, and name, Logarithms, was such as we have described. But it is unnecessary for us to take this remote though simple view of the matter. Instead of referring to a geometric and corresponding arithmetic series, we may immediately consider logarithms as the indices of any chosen number, r, whose powers are respectively equal to the digital numbers. For since $r^m \times r^n = r^{m+n}$, by ART. 36, AL. if we have a table containing the numbers to which r^m and r^n and r^{m+n} are respectively equal, with the indices m, n, $m+n$, opposite the numbers to which they belong, in order to multiply the two first numbers we need only add their indices, or logarithms, whose sum is the index, or logarithm, of the product sought, and therefore will be found over against it; looking down the second column till we meet this logarithm, we are accordingly directed to the product required. Thus, as we have already seen, if r be 10, the numbers 1, 2, 3, 4, &c. are equal respectively to $10^{0{,}000000}$, $10^{0{,}301030}$, $10^{0{,}477121}$, $10^{0{,}602060}$, &c. Consequently, registering these numbers and their corresponding logarithms, as in the Table aforegoing, we may proceed as above. All numbers are in this manner considered as the powers of some one number, r, and logarithms as the indices of this number *.

---

* We cannot choose r equal to 1, because every rational power of 1 is 1, and therefore cannot represent all numbers.

Definition. The *base* of a system of logarithms is that number whose different powers are made equal to the natural numbers.

Thus the base of the two systems preceding are respectively 10 and 2; the different powers of 10 in the one instance, and of 2 in the other, being made equal to the natural numbers, 1, 2, 3, 4, &c.

Obs. That system whose base is 10, being most in use, is designated the *common* system of Logarithms *.

Article 1. *In every system of logarithms, the log. of unity is cipher.*

For whatever r be, $r^0 = 1$, by *Obs.* Art. 37, Al.; that is, 0 is the log. of 1.

Thus in the common system, $10^0 = 1$; and also in the other system, $2^0 = 1$.

Art. 2. *In every system of logarithms, the log. of the base is unity.*

For whatever r be, $r^1 = r$; that is, 1 is the log. of r, the base.

Thus in the common system $10^1 = 10$; and also in the other system $2^1 = 2$.

Art. 3. *The log. of the product of any two numbers is equal to the sum of their logs.*

Let n, n', be any two numbers, l, l' their logarithms. Then log. $(n\,n') = l + l'$.

Demonstration. For if r be the base, then $r^l = n$, and $r^{l'} = n'$. Consequently

$$n\,n' = r^l \times r^{l'}.$$

But, by Art. 36, Al., $r^l \times r^{l'} = r^{l+l'}$.

Therefore, $$n\,n' = r^{l+l'}.$$

---

* This base was chosen by Mr. Briggs, professor of geometry at Gresham College. Baron Napier, or Neper, of Merchiston, in Scotland, was the inventor of the science, about 1614. He selected the base 2,302, &c. which is therefore called the *Neperian* base, and the system formed upon it the Neperian system of logarithms. From its connexion with a geometrical figure called the *Hyperbola*, this latter system is also called the *Hyperbolic*.

That is, $n\,n'$ being equal to *this* power of the base r, the log. of $n\,n'$ is $l + l'$; or, in other words, log. $(n\,n') = l + l'$. This, &c.

Examples of this have been given already. Thus, in the common system, where r is 10, log. $(2.5) =$ log. 2 + log. 5; as is evident from the Table, because log. (2.5) scil. 1,000000, is equal to log. 2, scil. ,301030 together with log. 5, scil. ,698970.

Obs. The assertion made in the above Article holds good for a product of any number of factors; thus, log. $(n\,n'\,n''$, &c.) = log. n + log. n' + log. n'' +, &c.

Art. 4. *The log. of the quotient of any two numbers is equal to the difference of their logs.*

Let n, n', be any two numbers, l, l' their logs. Then

log. $\left(\frac{n}{n'}\right) = l - l'$.

Dem. If r be the base, then $r^l = n$, and $r^{l'} = n'$. Consequently

$$\frac{n}{n'} = \frac{r^l}{r^{l'}}.$$

But, by **Art. 37, Al.**, $\frac{r^l}{r^{l'}} = r^{l-l'}$.

Therefore, $\frac{n}{n'} = r^{l-l'}$.

That is, $\frac{n}{n'}$, being equal to *this* power of the base r, the log. of $\frac{n}{n'}$ is $l - l'$; or, in other words, log. $\frac{n}{n'} = l - l'$. This, &c.

Thus, in the common system, where $r = 10$, log. $\left(\frac{8}{4}\right) =$ log. 8 — log. 4; as may be seen from the Table, by subtracting log. 4, which is .602060, from log. 8, which is .903090, because the remainder is .301030, which is the log. of 2, or $\frac{8}{4}$.

Obs. 1. From this Article we see that the work of *Division* may be performed by *Subtraction,* with the help of a logarithmic table: another faculty of logarithms most useful in computation.

Obs. 2. The assertion made in the last Article may be extended like that of ART. 3. Thus, $\log. \frac{n\,n'}{n''\,n'''} = (\log. n + \log. n') - (\log. n'' + \log. n''')$. And, in general, the log. of any number of factors divided by any other number of factors, is equal to the sum of all the logs. of the latter subtracted from the sum of all the logs. of the former.

ART. 5. *The log. of the power of any number is equal to the log. of the number multiplied by the index of the power.*

Let n^m be the power, and l the log, of any number n. Then $\log. (n^m) = l \times m$.

DEM. If r be the base,

$$n^m = (r^l)^m.$$

But, by *Obs.* ART. 38, AL. $(r^l)^m = r^{l.m}$.

Therefore, $n^m = r^{lm}$.

That is, n^m being equal to *this* power of the base r, the log. of n^m is $l.m$; or, in other words, $\log. n^m = l.m$. This, &c.

Thus, in the common system, $\log. (3^2) = \log. 3 \times 2$; as appears from the Table by taking out the log. of 3, scil. ,477121, and multiplying this by 2. The product is ,954243, which is the log. of 9, or 3^2; therefore the log. of $3^2 = \log. 3 \times 2$.

Obs. From this Article we see that *Multiplication* may be substituted for *Involution,* by help of a logarithmic table. In order to raise any number to any required power, we have only to multiply the log. of the simple number by the index of the required power; then looking down the column of logarithms, until we find one equal to this product, the opposite number is the power required. Thus to raise 2 to the 3d power: the log. of 2 is ,301030, and this multiplied by 3 gives ,903090; but as $\log. 2^3 =$

log. 2 × 3, this product, ,903090, must be the log. of the power required, and therefore stands opposite it in the table. Accordingly we find 8 to be the opposite number, which is evidently equal to the 3d power of 2.

Art. 6. *The log. of the root of any number is equal to the log. of the number divided by the index of the root.*

Let $\sqrt[m]{n}$ be the root, l the log. of any number n. Then $\log. \sqrt[m]{n} = \frac{l}{m}$.

Dem. If r be the base,

$$\sqrt[m]{n} = \sqrt[m]{r^l}.$$

But, by *Obs.* Art. 39, Al. $\sqrt[m]{r^l} = r^{\frac{l}{m}}$

Therefore, $\sqrt[m]{n} = r^{\frac{l}{m}}$.

That is, $\sqrt[m]{n}$ being equal to *this* power of the base r, the log. of $\sqrt[m]{n}$ is $\frac{l}{m}$; or, in other words, log. $\sqrt[m]{n} = \frac{l}{m}$. This, &c.

Thus, in the common system, log. $\sqrt[3]{8} = \frac{\log. 8}{3}$; as is plain from the Table, inasmuch as log. 8 is ,903090, and this divided by 3 gives ,301030, which is the log. of 2, or $\sqrt[3]{8}$.

Obs. This Article shows that the process of *Division* may be substituted for the more operose one of *Evolution,* by the assistance of a logarithmic table. To extract a required root of any number, it is sufficient if we divide the log. of that number by the index denominating this root; then the number in the table which stands opposite the quotient is the root required. Thus, to find the square root of 9: the log. of 9 is ,954243, and this divided by 2 gives ,477121; but as log. $\sqrt{9} = \frac{\log. 9}{2}$, this quotient, ,477121, must be the log. of the number required, which therefore will correspond to it in the Table. But the corresponding number to ,477121 is 3; and this is accordingly the square root of 9, which is found by a simple reference to the Table.

The above few Articles contain the chief properties of Logarithms; and with a knowledge of these the reader will be fully able to understand that part of any computation in which this instrument of science is employed.

ART. 6a. *The log. of a proper fraction is a subtractive number.*

Let $\frac{m}{n}$ be the fraction. Then log. $\left(\frac{m}{n}\right)$ is a subtractive number.

DEM. By ART. 4, log. $\frac{m}{n}$ = log. m — log. n. But the fraction being proper, m is less than n; therefore also log. m is less than log. n, and consequently log. m — log. n is, upon the whole, a subtractive number. This, &c.

Thus log. $\frac{3}{4}$ = log. 3 — log. 4; but log. 3 — log. 4 is, by the Table, equal to ,477121 — ,602060, which is a subtractive number, scil. — ,124939.

ART. 6s. *The log. of a fraction is equal to the log. of its reciprocal with a different sign.*

Let $\frac{m}{n}$ be the fraction. Then log. $\frac{m}{n}$ = — log. $\frac{n}{m}$.

DEM. Log. $\frac{m}{n}$ = log. m — log. n, by ART. 4.

log. $\frac{n}{m}$ = log. n — log. m, by the same.

But (log. m — log. n) = — (log. n — log. m); that is, log. $\frac{m}{n}$ = — log. $\frac{n}{m}$. This, &c.

Thus log. $\frac{3}{4}$ = — log. $\frac{4}{3}$. For log. $\frac{3}{4}$ = log. 3 — log. 4, *i. e.* by the Table, ,477121 — ,602060, or — ,124939; and log. $\frac{4}{3}$ = log. 4 — log. 3, *i. e.* by the Table, ,602060 — ,477121, or ,124939.

Obs. 1. In the common system of Logarithms, where the base is 10, the successive powers of 10, whose indices are additive integers, have for their logs. the successive digits; that is, the logs. of

1, 10, 100, 1000, 10000, 100000, 1000000, &c.

are, respectively,

0, 1, 2, 3, 4, 5, 6, &c.

The log. of any number between 1 and 10 will therefore be some decimal fraction between 0 and 1; the log. of any number between 10 and 100 will be some mixed number, made up of an integer and a decimal, between 1 and 2; the log. of any number between 100 and 1000 will be some mixed number of the same kind between 2 and 3; and so on. Hence, when we know between what two powers of 10 a number lies, we immediately know the *integral* part of its logarithm; for it is that of the lesser power. Thus, the integral part of log. 12 must be 1, inasmuch as 12, lying between 10^1 and 10^2, its log. must be greater than that of the former (scil. 1), but less than that of the latter (scil. 2).

Obs. 2. The integral part of a logarithm is called its *characteristic.*

Because it characterizes all the logarithms between two successive powers of the base.

Obs. 3. By ART. 6^{2} the logarithms of

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \frac{1}{1000000}, \text{ \&c.}$$

or, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, &c.

are respectively,

-1, -2, -3, -4, -5, -6, &c.

Hence, as logarithm $1 = 0$, the logarithm of any number between ·1 (or 10^{-1}) and 1, must be between -1 and 0; that is, it must consist of -1, and some *additive* decimal fraction, such that the whole logarithm will not be equal to 0. In the same way the logarithm of any number between 10^{-1} and 10^{-2}, must be between -2 and -1; that is, it must consist of -2, and some *additive* decimal which will not render the whole logarithm equal to -1: and so on. From this we see that the subtractive integers of logarithms are also characteristics; as they belong respectively to each class of logarithms, between the successive powers of 10 with subtractive indices. We cannot write down the logarithms of such powers with a subtractive sign *before* them, as it might sometimes be forgotten that this sign appertained solely to the characteristic integer, the decimal part of the log. being additive; we therefore write them down with a subtractive sign *above* the characteristic, which shows that this is subtractive, whilst the decimal part is additive. Thus, the log. of 1 is written $\bar{1}$·000000, which signifies — 1 + ·000000; the log. of ·46521, which lies between ·1 and 1, is written $\bar{1}$·667649, which signifies — 1 + ·667649.

NOTE. In some logarithmic Tables the characteristic is omitted, being so easily supplied by the reader, who has merely to consider between what two successive powers of 10 the number of which he seeks the logarithm occurs. Thus, log. 99 is sometimes given in the Tables ·995635, although it is truly 1·995635. But the reader considering that 99 lies between 10^1 and 10^2, knows that the characteristic, or integral part, of its logarithm must be 1, with which it is therefore needless to encumber the Table.

THE APPLICATION OF ALGEBRA TO GEOMETRY.

LESSON I.

Principles of the Science.

Those who have read our Algebra must have observed how much computation is abbreviated by the use of signs, or symbols; thus the formula

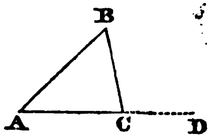
$$(a-b)^2 = a^2 + b^2 - 2ab$$

expresses in a very brief manner the following circumlocutory theorem: the *square* of the *difference* between *a* and *b is equal to* the *sum* of the *squares* of *a* and of *b, diminished* by *twice the product* of *a* and *b*. It must have been likewise sufficiently manifest that the operations performed on quantities, as well as the methods of expressing their mutual relations, were greatly facilitated by the same means. But the symbols +, —, =, are in reality no more proper to Algebra than to Geometry; they may serve to connect *any* quantities whatsoever which are capable of addition, subtraction, and equality. If, therefore, we find them so useful in the one Science, why not adopt them in the other? Why not apply them to geometrical as well as numerical quantities? Let us compare, for example, ART. 37, GEOMETRY, as given in that treatise, with the same Article, expressed in symbols.

ART. 37. *The three internal angles of any triangle taken together are equal to two right angles.*

In the triangle ABC, the angles ABC, BCA, CAB, taken together, are equal to two right angles.

Dem. By Art. 9, the angles BCA, BCD together are equal to two right angles; but by preceding Art. the angle BCD is equal to the two angles ABC, BAC together. Hence, the angles BCA, ABC, BAC together are equal to two right angles. This, &c.



The same in symbols:

Art. 37. *The 3 internal angles of any triangle together = 2 right angles.*

In the triangle ABC, the angles ABC + BCA + CAB = 2 right angles.

Dem. By Art. 9, the angles BCA + BCD = 2 right angles; but by preceding Art. the angle BCD = the angles ABC + BAC. Hence the angles BCA + ABC + BAC = 2 right angles. This, &c.

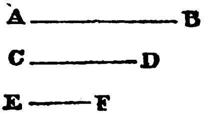
Here we see that the use of symbols enables us to contract our expressions and operations to a remarkable degree; and the abbreviation would be much more evident in a longer theorem. Henceforward, therefore, we shall adopt these signs wherever it is convenient. Unwillingness to burthen the student's memory with more than was absolutely requisite, alone prevented us from adopting them in our Geometry. Now that he has become familiar with them, he will find every demonstration in which they are used become not only shorter, but *clearer* by their introduction.

Again: The letters of the alphabet, a, b, c, &c., x, y, z, are not restricted by any necessity whatever to stand for numbers; they may represent whatever we like, and geometrical quantities with the same advantage as numerical. Thus, in the above triangle, the letter a may stand for the angle ABC, the letter b for the angle BCA, and the letter c for the angle CAB. If the letter d also stood for the angle BCD, and the letter r for a right angle, the above demonstration would appear under this still more abbreviated form.

Dem. By Art. 9, $b + d = 2r$; but by preceding Art. $d = a + c$. Hence, $b + a + c = 2r$. This, &c.

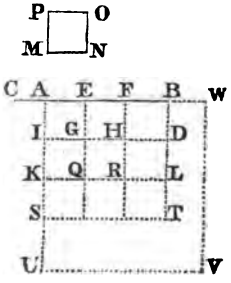
So far, and in similar cases, there is no difficulty nor impropriety, in the use of letters to represent geometrical quantities. But aware of the great facility afforded by the algebraical methods of computation, when we seek to apply these to Geometry, some difficulty and a great deal of impropriety would arise if the application were made directly, and without suitable preparation. Throughout Algebra, letters *must* represent numbers; because they are multiplied, divided, rooted, &c., which terms are properly numerical, and cannot be transferred, except in the way of analogy, to geometrical operations. If, therefore, we take certain letters to represent (as they may, without any preparation) certain geometrical, and likewise certain numerical quantities, which have no connexion with the former; if we then put these letters through a series of *algebraical* operations, these will be always rational when the letters are considered as numbers, but often totally absurd when the letters are considered as lines, angles, parallelograms, &c.

Thus, if a, b, c, stand respectively for the lines AB, CD, EF, and likewise for the numbers 6, 4, 3, respectively; then, $\frac{bc}{a}$ considered in its numerical capacity, is equal to a fourth-proportional to a, b, and c, by *Obs.* 2, Art. 3, Alg. i. e. $\frac{4\times 3}{6} = 2$, which is a fourth-proportional to 6, 4, and 3. But if a, b, and c be considered as lines AB, CD, and EF, then $\frac{bc}{a}$ will represent $\frac{\text{CD} \times \text{EF}}{\text{AB}}$, i. e. the *line* CD *multiplied* by the *line* EF, and this *product divided* by the *line* AB; which is wholly unintelligible.



But there is a mode of connecting geometrical with numerical quantities, so that the results of operations algebraically performed on the one should be applicable to the other; thereby enabling us, indirectly, to avail ourselves of algebraical computation in finding out the properties of lines, angles, and figures, which is the object of the Application of Algebra to Geometry.

As there does not subsist any natural connexion between the above two kinds of quantity, we must institute such a one as will serve our purpose. All *numbers* are referred to unity; and by their relation to it they are estimated. Now, suppose we take a line of a certain length, and to this refer all other lines, and estimate their magnitude by their relation to it, will not this chosen line be to all other lines what unity is to all other numbers? Exactly: and in consequence we may designate this particular line the *linear unit.*—If MN, for instance, be the linear unit chosen, then every other line may be considered as made up of it, or some submultiple of it, repeated a certain number of times, in the same manner as every other number is considered as made up of unity, or some submultiple of it, repeated a certain number of times*. Thus, suppose AB is *three* times the length of MN, this line AB will have the same ratio to MN as 3 has to 1. And if AC is equal to three-fourth parts of MN, this line AC will have the same ratio to MN as ¾ has to 1. Also AB will have to AC the same ratio as 3 has to ¾. We have now established a connexion between *lines* and numbers; there is little more difficulty in establishing one between *surfaces* and numbers.



Suppose a square described on MN, and the line AI taken

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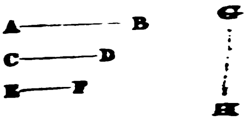
* There are certain numbers, *viz. surds*, or *incommensurable* numbers, which neither unity, nor any submultiple of it, how often soever repeated, equals exactly, but always leaves a remainder less than unity, or the submultiple taken. Hence, by taking a submultiple continually less and less, there will be left a remainder continually less and less; so that at length we may take such a submultiple of unity as will leave a remainder so small that the surd may be considered as made up of this submultiple a certain number of times repeated. There are also certain *lines* incommensurable with the linear unit, i. e. which neither it, nor any submultiple of it, how often soever repeated, equals exactly. But "in the same manner" we may take such a submultiple of the linear unit as will leave a remainder so small that the incommensurable line may be considered as made up of this submultiple a certain number of times repeated.

= MN, and the rectangle AIDB completed. It is evident that if the parallels EG, FH, be drawn, the several rectangles into which AIDB is divided are squares, each equal to MNOP, and as many in number as there are parts = MN in AB. Consequently the whole rectangle AIDB has the same relation to MNOP as the number of these parts has to unity, *i. e.* as 3 : 1. In the same way, if AK be taken = *twice* MN, and the rectangle AKLB be completed, and the parallels EQ, FR, be drawn, there will be *twice* as many squares = MNOP in the rectangle AKLB as there are parts = MN in AB, i. e. AKLB is to MNOP as 2 × 3 : 1: And so on; if AS be taken = *thrice* MN, there will be *thrice* as many squares = MNOP in the rectangle ASTB as there are parts = MN in AB, i. e. ASTB is to MNOP as 3 × 3 : 1. Generally: the rectangle AUVW is to MNOP as the *number* (suppose a) of MN's in AB, multiplied by the number (suppose b) of MN's in AU, is to unity, i. e. AUVW is to MNOP as $a \times b : 1$. Hence, as the square of MN has to any other rectangle under two lines, such as AW, AU, the same relation as unity has to the product whose factors represent these lines, we may designate MNOP the unit of rectangles; or more generally, the *superficial unit*, all surfaces being reducible to rectangles. Every surface may be considered as made up of MNOP, or some submultiple of it, repeated a certain number of times, in the same manner as every product is considered as made up of unity, or some submultiple of it, repeated a certain number of times. Thus we have established a connexion between *surfaces* and products.

It is to be observed, that in establishing the above connexion between products and rectangles, we always suppose the sides of the latter referred to that *linear* unit, whose square is the *superficial* unit to which we refer the rectangles. Thus, in the above figure, the rectangle AUVW is referred to the square MNOP as its unit, and the sides of this rectangle AU, AW, are each referred to a line MN as their unit, of which line MNOP is the square. This is much more convenient than to refer lines to a linear unit, such as MN, and surfaces to a superficial unit either greater or less than the square of MN, though we may take any linear and superficial units we please, however independent of each other. Algebra may now be applied to Geometry without



further reservation. If we have any lines AB, CD, EF, we may take any letters, a, b, c, to represent them; and running these letters through any suit of algebraical operations, the results found will primarily be those of numbers; but by means of the connexions above established, these results may be translated into geometrical language, so as to indicate corresponding results in the lines represented. Thus, if CD be a mean proportional between AB and EF, let a stand for AB, b for CD, and c for EF: then,

$$a : b :: b : c$$

$$\therefore \text{by Art. 8, Alg.} \quad a \times c = b^2;$$

that is, the product of the extremes is equal to the 2d power of the mean; which, translated into geometrical language, according to the connexion above established, is, that the *rectangle* under the extremes AB and EF is equal to the *square* of the mean CD. In this manner the algebraical result points out a geometrical truth (Art. 102, Geom.), not very remote, indeed, but sufficiently illustrative.

In all such calculations, therefore, we have only to keep in mind that the geometrical quantities represented by letters are related to a linear, or superficial, unit, as the number, or product, is related to 1. It is not, however, necessary to *specify* the geometrical unit; we have only to *suppose* such a unit, in order that the calculation should be applicable to Geometry. With the above recollection, in every such calculation, the first power of a letter is always to be understood as representing a line (referred to a linear unit as the letter itself is to 1); and a product of two letters is always to be understood as representing a rectangle (its sides referred to that linear unit, and its surface to the square of the same, as the factors and the product are to 1).

In the Application of Algebra to Geometry, the signs $=$, $\times$, $\div$, $\sqrt{\ }$, &c. have frequently a somewhat different meaning from their purely algebraical one; a meaning, however, dependent on the connexion above established, and intelligible therefrom.

1°. The sign of *equality* is often put between a letter and a geometrical quantity; not in its usual acceptation, as denoting that the former is equal to the latter, but that the former has to unity the same relation as the latter to the geometrical unit. Thus, $a =$ AB, in the above figure, means, not that a is equal to AB, which would be absurd, as a number cannot equal a line, but that $a : 1 :: \text{AB} : \text{MN}$ (the linear unit). So that the sign $=$, in such cases, must be translated by the word *represents,* instead of *is equal to;* $a =$ AB means that a *represents* the line AB.

2°. The sign of *multiplication* is often put between two lines; not in its usual acceptation, as denoting that these lines are to be multiplied, which would be absurd; but that the *number* of linear units in the one is to be multiplied by the *number* of linear units in the other; which will give the number of *superficial* units in the rectangle. In our Algebra, p. 43, we have shown, that from the analogy between a product and rectangle the geometrical term *square* was applied to the *second power* of a number; so, reciprocally, the numerical term *product* is, in this science, often applied to a rectangle under two lines, the lines being considered as the factors of the product. Henceforward, therefore, when we speak of the *product* of two lines, or put the sign × between them, thus, $\text{AB} \times \text{CD}$, it is to be understood that we mean only the *rectangle* under such lines, or the product of the *numbers* which express their respective ratios to the linear unit. For example, if $\text{AB} = 2\text{MN}$, and $\text{CD} = 3\text{MN}$, then $\text{AB} \times \text{CD} = 3\text{MN} \times 2\text{MN}$, that is, *six* times the *square* of MN, supposing the lines formed into a rectangle.

3°. The sign $\div$ between two lines, thus, $\text{AB} \div \text{CD}$, is not recognised in the Application of Algebra to Geometry; but thus, $\frac{\text{AB}}{\text{CD}}$, it signifies the same as it does in Algebra, namely, *a ratio,* viz. that of $\text{AB} : \text{CD}$, just as the numbers a and b in this form $\left(\frac{a}{b}\right)$ express the ratio of a b, (ART. 10. ALG.). Almost all the rules of Fractions will be found applicable to such *geometrical* fractions referred to their proper unit, according to the system above explained.

4°. The sign $\sqrt{\quad}$, over a number, is used to express that other number whose second power is the given number; thus, $\sqrt{a^2}$ is another way of expressing a whose square is a^2. And in like manner the sign $\sqrt{\quad}$, over a geometrical quantity, is used to express that other geometrical quantity whose *square* is the given geometrical quantity; thus, $\sqrt{\text{AB}^2}$ is another way of expressing the line AB whose square is AB^2. Likewise $\sqrt{a \times b}$ designates that number (suppose x) whose 2d power $= a \times b$, for $\sqrt{ab} = \sqrt{x^2} = x$; and $\sqrt{\text{AB} \times \text{CD}}$ designates that line (suppose GH) whose square $= \text{AB} \times \text{CD}$, for $\sqrt{\text{AB} \times \text{CD}} = \sqrt{\text{GH}^2} = \text{GH}$. And so on for all other quantities under the radical sign. When, in the present science, therefore, we speak of the square-*root* of a square, or rectangle, we mean in the first case the *side* of the square, in the second case the *side* of a square equal to the rectangle [*].

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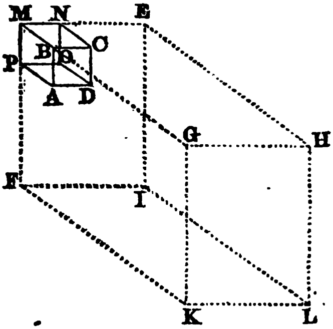
[*] It is easy to extend the above principles to solids:



A cube is a solid contained by six equal squares, as in the annexed figure, where MNOP, MPAB, MBCN, ADCB, DCNO, DOPA, are all squares equal to each other.

Now, if MN, MP, MB, be produced to any lengths ME, MF, MG, and the rectangles MEIF, MFKG, MGHE, be completed; also if the planes GKLH, LHEI, ILKF, be respectively parallel to these rectangles, then the solid FIEMGHLK is called a *rectangular parallelopiped*. Hence, taking MN as linear unit, the cube MNOPABCD may be considered as the *solid unit;* and every other rectangular parallelopiped, as that in the figure, may be considered as made up of this solid unit, or some submultiple of it, repeated a certain number of times: Consequently, if MN be represented by 1, and ME by 3, then the parallelopiped under ME, MP, and MB, will be thrice the solid unit, or will be to it as 3 : 1; and if MF be represented by 4, then the increased parallelopiped under ME, MF, and MB, will be 4 times the other, or 12 times the solid unit, i. e. will be to it as 3 × 4 : 1; and, finally, if MG be represented by 5, then the still-increased parallelopiped under MG, MF, and ME, will be 5 times the last, or 60

LESSON II.

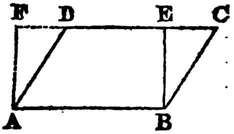
Practical Illustration of the Science.

In the foregoing observations are contained the principles of this Science; its practice may be illustrated by a very few examples, which we shall choose for their prospective utility in other parts of our Series, and generally in all mathematical investigations. As we shall have to refer to them, it may be well to class them as *Articles*.

Definition. The *Area* of a figure is the portion of space contained within its boundaries.

Article 1. *The area of a parallelogram is equal to the product of any side taken as base, and the corresponding altitude.*

In the parallelogram ABCD, if the side, AB, be taken as base, and the perpendicular to it, BE, as altitude; then, the area of ABCD = AB × BE.



Dem. Complete the rectangle ABEF, which is = ABCD by Art. 28, Geom. But, by preceding observations, the area of the rectangle ABEF is expressed by multiplying the number of linear units in AB into the number of linear units in BE; that is, by AB × BE, supposing each line made up of the linear unit (or some submultiple of it) repeated. Therefore also the area of its equal, the paral-

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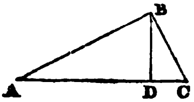
times the solid unit, i. e. will be to it as $3 \times 4 \times 5 : 1$; and so on. In general, the parallelopiped is to the cube as the *number* (suppose a) of MN's in ME, multiplied by the number (suppose b) of MN's in MF, multiplied again by the number (suppose c) of MN's in MG, is to unity, i. e. the parallelopiped is to the solid unit as $a \times b \times c : 1$.

Thus, a rectangular parallelopiped (to which all solids are reducible) may be represented algebraically, by a product of three factors representing severally the three adjacent edge-lines of the parallelopiped; and whenever we meet such a product it is to be considered geometrically as a parallelopiped whose adjacent edge-lines are to the linear unit as the factors of this product are to unity. Hence it follows that the product $a \times a \times a$, or a^3, represents a *cube* in geometry; and, on the contrary, that a cube is represented geometrically by the third power of that number which represents any one of its edge-lines, according to what we have said, p. 44, Algebra.

lelogram ABCD, is expressed by AB × BE; or in symbols, ABCD = AB × BE. This, &c.

Thus if AB were 6 times MN (the linear unit *), and EB 3 times the same, then the area of ABCD would be = 18MN^2.

ART. 2. *The area of a triangle is equal to half the product of any side taken as base, and the corresponding altitude.*

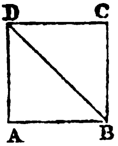


In the triangle ABC, if the side AC be taken as base, and the perpendicular to it, BD, as altitude; then, the area of ABC $= \dfrac{\text{AC} \times \text{BD}}{2}$.

DEM. By ART. 31, GEOM., the triangle ABC is equal to half the parallelogram on the same base and between the same parallels (i. e. having the same altitude). But by ART. preceding, a parallelogram on the base AC with the altitude BD would be = AC × BD. Hence, the triangle ABC is $= \dfrac{\text{AC} \times \text{BD}}{2}$. This, &c.

Thus if AC were 8 times MN, and BD 4 times the same, then the area of the triangle ABC would be = 16MN^2.

ART. 3. *The side of a square is to its diagonal as unity to the square-root of two.*



Let ABCD be a square. Then, AB : BD : : 1 : $\sqrt{2}$.

DEM. Represent AB by s, and BD by d. Therefore, ART. 47, GEOM., as AB = AD

$$d^2 = 2s^2$$

$$\therefore \quad d = \sqrt{2} \,.\, s$$

$$\therefore \text{ ART. 7, ALG.} \quad s : d :: 1 : \sqrt{2},$$

$$\text{i. e.} \quad \text{AB} : \text{BD} :: 1 : \sqrt{2}.$$

This, &c.

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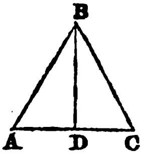
[*] In practice the linear unit is generally supposed to be a line of an *inch* in length. So that in this case the area of a parallelogram whose base was 6 inches, and altitude 3, would be = 18 squares of an inch line, i. e. 18 *square-inches*.

Art. 4. *The altitude of an equilateral triangle is to the side as the square-root of three to two.*



Let ABC be an equilateral triangle, BD its altitude. Then, BD : AC :: $\sqrt{3}$: 2.

Dem. Represent BD by p, and AC by s; therefore, as AC is divided at the point D into two equal parts (Art. 139, Geom.) either of them AD, will be represented by $\frac{s}{2}$. But, by Art. 47, Geom., $AB^2 = BD^2 + AD^2$, that is,

$$s^2 = p^2 + \left(\frac{s}{2}\right)^2$$

$$\therefore p^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4},$$

$$\therefore p = \frac{\sqrt{3}\,.\,s}{2}$$

$$\therefore 2p = \sqrt{3}\,.\,s$$

$$\therefore p : s :: \sqrt{3} : 2$$

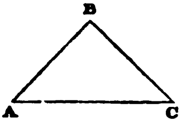
This, &c.

Observation. The altitude of an equilateral triangle is to half the side as the square-root of three to unity.

For, by the third equation above,

$$p : \frac{s}{2} :: \sqrt{3} : 1.$$

Art. 5. *Given any two sides of a right-angled triangle, the third may be found.*



Let any two sides of the triangle ABC, right-angled at B, be given, or known. Then, the third remaining side can be found.

Dem. Represent the three sides AB, BC, CA, by the letters s, s', s'', respectively.

By Art. 47, Geom.,

$$s''^2 = s^2 + s'^2;$$

therefore, any two of these quantities being given the third will be determined. First let the sides s, s', *about* the right angle be given; then,

$$s'' = \sqrt{s^2 + s'^2}.$$

that is, the third side CA is equal to the *side* of a square equal to the sum of squares of AB and of BC. Again, let the side, s'', *opposite* the right angle, and either of the other sides, s, be given; then,

$$s'^2 = s''^2 - s^2$$
$$\therefore\ s' = \sqrt{s''^2 - s^2},$$

that is, the third side BC is equal to the *side* of a square equal to the difference of the squares of CA and of AB. This, &c.

Thus, if the sides, s, s', about the right angle of a triangle were given respectively equal to 6 and 8 times the linear unit (an inch, for example, which we will represent by i), then,

$$\begin{aligned} s'' &= \sqrt{(6i)^2 + (8i)^2} \\ &= \sqrt{36\,i^2 + 64\,i^2} \\ &= \sqrt{100\,i^2} = 10\,i; \end{aligned}$$

that is, the third side in this case would be equal to *ten* times the linear unit, or 10 inches.

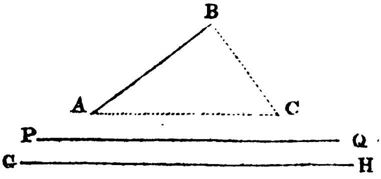
Again, if the side, s'', opposite the right angle of a triangle were given equal to 5 times i, and one of the other sides, s', equal to 4 times i; then,

$$\begin{aligned} s^2 &= \sqrt{s''^2 - s'^2} \\ &= \sqrt{(5\,i)^2 - (4\,i)^2} \\ &= \sqrt{25\,i^2 - 16\,i^2} \\ &= \sqrt{9\,i^2} = 3\,i; \end{aligned}$$

that is, the third side in this case would be equal to *three* times the linear unit, or 3 inches.

ART. 6. *In a right-angled triangle, given one of the sides about the right angle, and the sum of the other two, the triangle itself may be found.*

Let ABC be a triangle right-angled at B. Then, if we are given one of the sides, as AB, about the right angle, and a right line PQ equal to AC + BC, we may find the triangle ABC.



Dem. Represent AB by s, and PQ by a, and AC (which is unknown) by x. Consequently BC $=$ PQ $-$ AC $= a - x$. Therefore, by Art. 47, Geom.

$$\begin{aligned} x^2 &= s^2 + (a - x)^2 \\ &= s^2 + a^2 + x^2 - 2ax \\ \therefore\ 0 &= s^2 + a^2 - 2ax \\ \therefore 2ax &= s^2 + a^2 \\ \therefore\ x &= \frac{s^2 + a^2}{2a} \end{aligned}$$

that is, if we take a right line (suppose GH represented by b) whose square $=$ AB^2 $+$ PQ^2 $= s^2 + a^2$, then

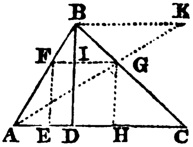
$$x = \frac{b^2}{2a}$$

or, in other words, x is a third proportional to $2a$ and b.

Having thus found AC, we may find BC, for BC $=$ PQ $-$ AC.

This, &c.

Art. 7. *Given the base and altitude of a triangle, the inscribed square may be found.*



Let ABC be a triangle, AC its base, BD its altitude. Then the area of EFGH, the inscribed square, may be found.

Dem. Represent the base AC by b, and the altitude BD by a, and GF the side of the inscribed square (which is unknown) by x. Then, Art. 112, Geom.

$$\begin{aligned} &\text{AC} : \text{FG} :: \text{BC} : \text{BG} :: \text{BD} : \text{BI}, \text{ or, as BI} = \text{BD} - \text{ID}, \\ &b : x \qquad :: \qquad a : a - x, \\ \therefore\ & b\,(a - x) = ax \\ \therefore\ & ba - bx = ax \\ \therefore\ & ba = ax + bx \\ \therefore\ & x = \frac{ab}{a + b} \end{aligned}$$

that is, x is a fourth proportional to $(a + b)$, b, and a. Having thus found the side of the inscribed square, the square itself is determined. This, &c.

Obs. If the question proposed be *numerical*, that is to say, if the known quantities be given as so many parts of an inch (an inch being supposed the linear unit), and if the sought quantity be required only in the same terms, this is immediately presented by the final equation. Thus, in the above example, if b were given $= 12$ inches, and $a = 4$ inches, then we should have

$$x = \frac{4 . 12}{4 + 12} = 3;$$

that is, the side of the inscribed square would be $= 3$ inches, and $\therefore$ the square itself would be $= 9$ square inches.

But if the question proposed be *geometrical*, that is to say, if the sought quantity be required in its *geometrical* magnitude, although its value is presented by the final equation, the quantity *itself* remains to be found by a construction. Thus, in the same example, if the square be required, not in numbers of square inches, but in actual geometric dimensions, we must, from the value found for x, scil.

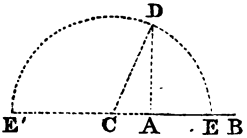
$$\frac{ab}{a + b},$$

construct it. And a method of doing this is suggested by the equation. Draw BK parallel to AC, and equal to BD. Join AK, and draw GF parallel to AC, from the point G, where AK meets one side of the triangle, to the other. Then, GF will be a side of the inscribed square. For, GF : BK :: GA : KA, by ART. 112, GEOM.. But the triangles AGC, and BGK, being equiangular to each other, AG : GK :: AC : BK, by the same ART. Therefore, *convertendo*, AG : AG + GK :: AC : AC + BK, or AG : AK :: AC : AC + BD. Consequently, GF : BK :: AC : AC + BD; that is, $x : a :: b : a + b$. Hence, x has been constructed a fourth-proportional to $(a + b)$, b, and a; and therefore is the side of the square required, on which we have only to form a square that the question may be resolved.

However, not only the actual geometric dimensions, but the due *position* of the sought quantity, may be required. This also is frequently indicated by the equation. Thus, in the same example, not only the size of the inscribed square may be required, but its due position. In order to investigate its due position, it is not sufficient that we find a fourth proportional to $(a + b)$, b, and a, for that will only be the side of the square, *in magnitude;* it might be found as well entirely apart from the figure as in it. But the construction given above, connected as it is with the lines of the figure, determines not only the magnitude of the square, but the point G, in which two of its sides, GF and GH, meet; so that the position, EFGH, of the square, is also determined. For, after having drawn GF as above, if we draw GH, FE, perpendicular to AC, it may be easily shown that the figure EFGH is a square, as follows. By ART. 112, GEOM. FE : BD :: FA : BA :: FG : BK. Therefore FE : BD :: FG : BK. But BD = BK, therefore FE = FG. Hence EFGH being a right-angled parallelogram, with two adjacent sides equal, is a square (DEF. XIII. GEOM.). In this manner we have found the actual square inscribed in the triangle, and its actual position.

Another example or two will elucidate these observations.

ART. 8. *To divide a right line into two such parts that the square of one may be equal to the rectangle under the other and the whole line.*



Let AB be the given right line. It is required to divide it into two parts, AE and EB, so that we may have $AE^2 = EB \times AB$.

OPERATION. Represent AB by a, and the unknown quantity AE by x. Then, as the square of the sought part AE must be equal to the rectangle $(AB - AE) \times AB$, we have this equation,

$$x^2 = (a - x)\,a$$

$$\therefore\ x^2 + ax = a^2$$

$$\therefore\ x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + a^2}$$

This equation determines the length of AE in numbers, at once; that of AB being supposed known. But if it be required to find the geometrical length of AE, we must *construct* the value of x presented by the equation.

CONSTRUCTION. Produce AB through A, and take $AC = \frac{AB}{2}$. Draw AD perpendicular and equal to AB. Join CD, and with CD as radius describe the semicircle EDE′ from the point C as centre. Then, the part AE cut off from AB by this semicircle will be the line sought.

DEM. By ART. 47, GEOM. $CD^2 = CA^2 + AD^2 = CA^2 + AB^2 = \left(\frac{AB}{2}\right)^2 + AB^2 = \frac{AB^2}{4} + AB^2$. Consequently, as $CD = CE$, we have

$$CE^2 = \frac{AB^2}{4} + AB^2$$

$$\therefore \quad CE = \pm\sqrt{\frac{AB^2}{4} + AB^2}$$

$$\therefore \quad CA + AE = \pm\sqrt{\frac{AB^2}{4} + AB^2}$$

$$\therefore \quad AE = -CA \pm\sqrt{\frac{AB^2}{4} + AB^2}$$

or,

$$AE = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + A^2}$$

Hence, we see that, by the preceding construction, AE has the same value as x, and is therefore the sought quantity, or the part of the line required *.

Obs. 1. In the application of Algebra to the solution of geometrical questions, we sometimes, as in the above example, obtain more than a single value for the sought quantity. Of these we must select that which will answer the question proposed; but the other will always answer some *other* question, related to the given one, and, indeed, merely an extension of it. Thus, inasmuch as we require the line AB, *itself*, to be divided as above, we must take the first value of x; for then

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* From the equation $x^2 = (a - x)\,x$, we have found the value of x, and therefore need only show that by our construction this value is represented geometrically in the line AE, as we have done above. For, as the value of x must be the true answer to the question, its representative must likewise be the true answer to the question. But we may demonstrate *geometrically*, that AE is the true answer to the question, thus: $CA^2 + AB^2 = CA^2 + AD^2 = CD^2 = CE^2 = CA^2 + AE^2 + 2CA \times AE$ (ART. 92, GEOM.). Consequently, $AB^2 = AE^2 + 2CA \times AE = AE^2 + AB \times AE$. Therefore, $AE^2 = AB^2 - AB \times AE = AB \times EB$ (ART. 90, GEOM.). This, &c.

$$x = +\sqrt{\frac{a^2}{4} + a^2} - \frac{a}{2} = \sqrt{\frac{\mathrm{AB}^2}{4} + \mathrm{AB}^2} - \frac{\mathrm{AB}}{2}$$

$= \mathrm{CE} - \mathrm{CA} = \mathrm{AE}.$

But the other value of x, scil.

$$x = -\frac{a}{2} - \sqrt{\frac{a^2}{4} + a^2} = -\frac{\mathrm{AB}}{2} - \sqrt{\frac{\mathrm{AB}^2}{4} + \mathrm{AB}^2}$$

$= -\mathrm{CA} - \mathrm{CE} = -(\mathrm{CA} + \mathrm{CE}) = -\mathrm{AE}',$

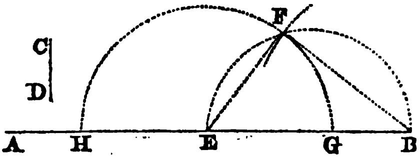
which has been found by the same construction, will answer a question different from, yet related to, the first; viz. To *produce* a right line, so that the square of the part produced may be equal to the rectangle under the line itself and the whole produced line.

Thus, AB has been produced through A to E′, and $\mathrm{AE}'^2 = \mathrm{E'B} \times \mathrm{AB}$, as the reader may easily demonstrate *. Hence, the application of Algebra to Geometry not only solves the question proposed, but by sometimes furnishing more than one answer, teaches us that our question was capable of more generality than we imagined, and enlarges our knowledge, by the suggestion of other truths which we did not look for.

Obs. 2. To divide a right line, as in the preceding Article, is technically denominated,—cutting a right line in *extreme and mean ratio;* because AE is a geometric mean between the extremes AB, EB; and AE′ a geometric mean between AB, E′B; by ART. 103, GEOM.

ART. 9 *Given the greater of two geometric extremes, and the difference between the lesser and mean, the mean may be found.*

Let AB be the greater extreme, CD the difference between it and the lesser. Then the mean may be found.



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* $\mathrm{AE}' = \mathrm{CE} + \mathrm{CA} = 2\mathrm{CA} + \mathrm{AE} = \mathrm{AB} + \mathrm{AE}$. Therefore, $\mathrm{AE}'^2 = \mathrm{AB}^2 + 2\mathrm{AB} \times \mathrm{AE} + \mathrm{AE}^2 = \mathrm{AB}^2 + 2\mathrm{AB} \times \mathrm{AE} + \mathrm{AB} \times \mathrm{EB} = \mathrm{AB} \times (\mathrm{AB} + 2\mathrm{AE} + \mathrm{EB}) = \mathrm{AB} \times (\mathrm{AB} + \mathrm{AB} + \mathrm{AE}) = \mathrm{AB} \times (\mathrm{AB} + \mathrm{AE}') = \mathrm{AB} \times \mathrm{E'B}$. This, &c.

Operation. Represent AB by a, CD by d, and the sought mean by x. By the condition of the question proposed, $\frac{x^2}{a}$ is the lesser extreme; and we have

$$x - \frac{x^2}{a} = d$$

$$\therefore\ ax - x^2 = ad$$

$$\therefore\ x^2 - ax = -ad$$

$$\therefore\ x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - ad}.$$

This equation shows that the question admits of two answers, though but one was sought. If, for instance, the lines were given in numbers, viz. AB = 6 (times the linear unit) and CD = $\frac{5}{6}$ (of the linear unit), then we should have

$$\begin{aligned} x &= 3 \pm \sqrt{9 - 6 . \tfrac{5}{6}} \\ &= 3 \pm \sqrt{4} \\ &= 5 \text{ or } 1; \end{aligned}$$

that is, the mean would be 5 times, or once, the linear unit; and the series of proportionals would be, in linear measure, either

$$6,\ 5,\ 4\tfrac{1}{6}.$$
$$\text{or, } 6,\ 1,\ \tfrac{1}{6}.$$

But if the geometrical answers be sought, we must exhibit them as follows:

Cons. Divide AB into two equal parts at E, and on either, as EB, describe the semicircle EFB. Find a mean proportional BF between AB and CD, by Prob. XX, Geometry. With B as a centre, and with the line BF as distance, describe an arch cutting EFB in F. Join EF, and with E as centre, and EF as distance, describe the semicircle HFG. Then, AG or AH, will be the mean sought.

Dem. By Art. 72, Geom. the angle at F is a right one; ∴ by Art. 47, Geom.

$$EF^2 = EB^2 - FB^2$$

$$\therefore\ EF = \pm \sqrt{EB^2 - FB^2}$$

$$\therefore\ AG = AE + EG = AE + EF = AE + \sqrt{EB^2 - FB^2},$$

and, $\text{AH} = \text{AE} - \text{EH} = \text{AE} - \text{EF} = \text{AE} - \sqrt{\text{EB}^2 - \text{FB}^2}$.

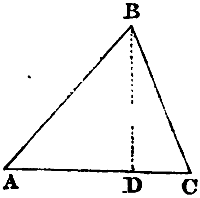
or, as $\text{FB}^2 = \text{AB} \times \text{CD}$, $\quad \text{AG} = \frac{a}{2} + \sqrt{\frac{a^2}{4} - ad}$.

and, $\text{AH} = \frac{a}{2} - \sqrt{\frac{a^2}{4} - ad}$.

Therefore AG and AH are the means, determined by the equation; and the series of proportionals are either

AB, AG, AG — CD,
or, AB, AH, AH — CD.

ART. 10. *Given the three sides of a triangle, we may find its area.*



Let ABC be any triangle, of which we are given the sides AB, BC, and AC. Then the area of the triangle may be found.

OPERATION. Suppose a perpendicular BD drawn to any side AC, from the vertex of the opposite angle. Represent AC by a, AB by b, BC by c; BD (which is unknown) by x, and AD, one of the segments of AC made by the perpendicular (which is also unknown), by y. Then, by ART. 2, the area of the triangle is $= \frac{xa}{2}$. But by ART. 47, GEOM.

$$b^2 = x^2 + y^2;$$
$$\text{and, } c^2 = x^2 + (a-y)^2.$$

Here we have two equations, with two unknown quantities; from which we can find the value of either, as y. Thus:

Obs. In Article 9, if ad be greater than $\frac{a^2}{4}$; that is, if the rectangle AB × CD be greater than the square of half AB, the values for the sought mean will be *impossible* (*Obs.* 2°, ART. 55, ALG.). In other words, no mean can be found under the conditions of the question. But the impossible form under which the root x would in such case appear, has the same use in this science that it has in pure Algebra. First: it teaches us how to modify the question, so that the whole quantity under the from the first, $x^2 = b^2 - y^2$,

. . . second, $x^2 = c^2 - (a-y)^2$,

$$\therefore \quad b^2 - y^2 = c^2 - (a-y)^2 = c^2 - a^2 - y^2 + 2ay,$$

$$\therefore \quad b^2 - c^2 + a^2 = 2ay,$$

$$\therefore \quad y = \frac{a}{2} + \frac{b^2 - c^2}{2a}.$$

The value of y is here determined in known quantities, a, b, and c. We have therefore only to subtract the square of this value from b^2, and we get x^2 in known quantities, by the first equation. The square-root of this will represent the perpendicular, which we are to multiply by the side on which it falls, and half the product will be the area required.

Thus, if AC, AB, and BC, were given respectively equal to 11, 20, and 13 times the linear unit; then we shall have

$$y = \frac{11}{2} + \frac{20^2 - 13^2}{2 \,.\, 11}$$

$$= \frac{11}{2} + \frac{21}{2} = 16.$$

Subtracting 16^2 from 20^2 we get 12^2, which is the square of the perpendicular. Therefore the perpendicular itself is 12; and half the product of this and the corresponding side 11, gives 66, which is the area of the triangle in squares of the linear unit.

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radical sign shall become additive, or at least $= 0$. Thus, if CD were given greater than $\frac{AB}{4}$, suppose $\frac{AB}{3}$, the radical quantity above would become

$$\sqrt{\frac{a^2}{4} - \frac{a \,.\, a}{3}},$$

which is impossible. This teaches us that CD must be given either $= \frac{AB}{4}$, or less than $\frac{AB}{4}$, in order that a mean should be found between AB and a line less than the mean by the quantity CD *.

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* It is to be noted, that if the series of proportionals is supposed an

LESSON III.

Obs. We will conclude this Treatise with a few general rules, by which the practitioner may find it useful to guide himself in the application of his algebraical knowledge to questions in Geometry. We shall, however, premise, that his own ingenuity and experience will suggest much better rules than we can pretend to give on this subject, which is one peculiarly dependent on such qualifications. Our intention is merely to assist him in that stage of his progress towards the full attainment of the science, which must be passed before ingenuity can be displayed, or experience obtained.

Secondly: it teaches us the limits between which the known quantities may vary. In the above question, for example, supposing AB of a determinate length, CD may be of any length between 0 and $\frac{AB}{4}$. Likewise, supposing CD of a determinate length, AB may be of any length greater than 4CD. That is to say, the least and greatest limits, or the *minimum* and the *maximum* of CD are, respectively, 0 and $\frac{AB}{4}$; the *minimum* of AB is 4CD, but it has no *maximum*, i. e. it may be given as great as we please.

In the same manner, whenever, by the application of Algebra to any geometrical question, we get an impossible form of root, it enables us to determine the *maxima* and *minima* of the quantities under consideration; and thus becomes of signal use in Geometry as well as Algebra.

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increasing one; that is, if the latter extreme exceeds the mean, we shall have, instead of $x - \frac{x^2}{a} = d$,

$$\frac{x^2}{a} - x = d$$

$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + ad},$$

and the answer will be *always* possible.

When, by proceeding on our hints, he becomes sufficiently master of the art to strike out new methods for himself, and to invent new expedients for the easy solution of questions placed before him, he may safely discard or neglect ours.

Rule I. When a geometric question is proposed to be resolved algebraically, a figure is to be drawn which shall represent all the several lines, angles, &c., &c., as well unknown as known, involved in the question; and this figure, with all its parts, is to be regarded as if it and all its parts were drawn as they should be if the question were already solved.

Thus, in Art. 6, the triangle ABC was drawn, and with its sides AC, BC, was regarded as if already found.

Rule II. If the lines drawn as above should not appear sufficient, or sufficiently convenient to an easy solution of the question, such other lines are to be drawn as may appear to facilitate this object, by their connexion with the known parts of the figure.

Thus, in Art. 10, to draw the triangle ABC was not sufficient; in order to determine its area, it was found convenient to draw the perpendicular BD, dividing the base into segments, from whose relation to the known sides BA, BC, we were enabled to resolve the question.

Rule III. Represent the several parts of the figure which may enter into calculation by letters; and after having duly considered the nature of the question, "state the several dependencies, or connexions, of these quantities, known and unknown, by means of these algebraical quantities, in as many separate equations; so that, if possible, the number of independent equations may be equal to the number of the unknown quantities," according to Art. 58, Algebra. We have then only to find the value, or values, of the unknown quantity, or quantities, and the proposed question will be resolved numerically, while the answer in almost every case will point out a geometrical construction.

Thus, in Art. 10, we represented the three sides AC, AB, BC, the perpendicular BD, and the segment AD, by the letters a, b, c, x, y, respectively; we then stated the connexions of these five geometric quantities by means of the five corresponding algebraic quantities, in as many independent equations as there were *unknown* quantities, *viz.* two;

$$b^2 = x^2 + y^2,$$
$$\text{and, } c^2 = x^2 + (a - y)^2.$$

which equations we were enabled to institute by duly considering the nature of the question proposed. We then found the value of y, by the common principles for the resolution of equations, and determined from it the value of x, by means of the first equation; this value for x, multiplied by half b, gave the area of the triangle, as required.

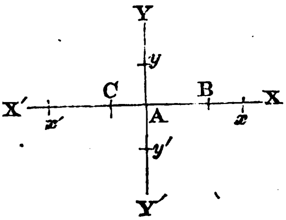
Obs. 1. The utility of drawing the figure, and all its parts, known and unknown, together with, on some occasions, additional lines, consists in this: that regarding the figure, with all its parts, as if it and they were all drawn as they should be, were the question already solved, this diagram will then suggest what relations *would* subsist between the known and unknown parts, if the latter were actually found. So that they may be actually found by stating those relations, and arguing upon them.

Thus, in ART. 6, by drawing the triangle ABC, and regarding AC, BC, as if they were already found, we perceive that the following relation subsists between AC, BC, and AB, viz. $AC^2 = BC^2 + AB^2$; and as, by the condition of the question, $BC = PQ - AC$, we have $AC^2 = (PQ - AC)^2 + AB^2$; that is, in algebraical language, $x^2 = (a - x)^2 + b^2$. By operating on this relation we obtain the value of x.

There are other Rules for the application of Algebra to Geometry; but as they contemplate a much greater advance in the study than is necessary for *our* readers, their insertion here would be superfluous.

Obs. 2. Before putting the above rules in practice, it may be useful to understand a method by which geometers agree to represent algebraical quantities with *different signs*, but all standing for right lines, in a way commodious for their own science.

If we take a right line X′X, and a point in it, A; and if we suppose all the *positive* or *additive* numbers, which, in an algebraical computation, represent lines, measured from the point A in one direction AX, then all the *negative* or *subtractive* numbers may be conveniently represented by lines measured in the *opposite* direction from A on AX′. Thus, if the portion Ax be taken to represent $+ a$, then an equal portion Ax', in the *other* direction, is considered to represent $- a$. Or, if YY′ be the measuring right line, and if the portion Ay be taken to represent $+ b$, then an equal portion Ay', in the *other* direction, is considered to represent $- b$.



The same method is pursued when, instead of algebraical quantities, right lines themselves occur with the sign + or − before them, signifying that such lines are to be added or subtracted. Thus, + AB is taken in one direction, and − AC in the opposite. We shall find the use of this principle in our next treatise.

PRAXIS.

1. What is the area of an equilateral triangle, in terms of its side s?

Answer. $\frac{\sqrt{3}}{4} \cdot s^2$.

2. In a right-angled triangle, given one of the sides (s) about the right angle, and the difference (d) of the other two, what are the values of the latter?

Ans. $\frac{s^2 - d^2}{2d}$; and $\frac{s^2 + d^2}{2d}$.

3. Given the base, b, of an equilateral triangle, what is the length of the side of the inscribed square?

Ans. $\frac{\sqrt{3}}{2 + \sqrt{3}} \cdot b$.

4. In a right-angled triangle whose sides about the right angle are equal, what is the length of the side of the inscribed square standing on either of them (s); and of that standing on the third side (s')?

Ans. $\frac{s}{2}$; and $\frac{s'}{3}$.

5. If an equilateral triangle be inscribed in a circle, whose radius is r, what is the length of its side?

Ans. $\sqrt{3} \, . \, r$.

6. In a right-angled triangle, given the side (h) opposite the right angle, and the ratio ($a : b$) of the other sides; what are the values of the latter?

Ans. $\frac{ah}{\sqrt{a^2+b^2}}$; and $\frac{bh}{\sqrt{a^2+b^2}}$.

TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

Preliminary.

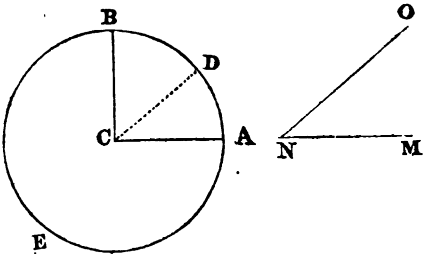
IN the Greek language, Τριγωνὸς (*Trigonos*), means a triangle, from the two words τρὶς (*tris*), three, and γωνία (*gonia*), an angle, because it has three angles; also Μέτρον (*Mètron*), signifies a measure: Hence the whole term TRIGONOMETRY signifies, in its primitive acceptation, the measurement of triangles. There are six, or more properly seven, parts in a rectilineal triangle, viz. the three sides, the three angles, and the area, or space included by the three sides. Certain of these parts being given, that is, being assigned in magnitude or proportion; the others, and therefore the triangle itself, have a relative magnitude in proportion; to compute which is the first though not the sole object of Trigonometry. Thus, if the sides about the right angle of a plane right-angled triangle be given each two feet in length; then, as these sides are equal, each of the acute angles must be half a right angle, by ART. 137, GEOMETRY; and, by ART. 47, GEOM. the third side may be discovered, because it is equal to a right line whose square equals the sum of the squares of the given sides, or twice the square of one of them. From one angle and two sides being given, we may thus determine the remaining angles and side: the area is likewise evidently half the square of either given side, and so we obtain a complete knowledge of the triangle. By methods not very dissimilar, but a little more difficult, we may compute the magnitude and form of any triangle, or of its parts, from being given certain of these parts, or certain relations between them.

This science is, however, often employed to compute the magnitude of angles alone, without considering them to belong to triangles; and instead of *Trigonometry*, it is then more properly called GONIOMETRY, or the measurement of *angles*. It will be more elementary to begin our Treatise with the principles of simple Goniometry, and then proceed to those of Trigonometry.

LESSON I.

Definitions, &c.

Observation 1. To determine the magnitude of any given angle, we have only to fix upon a certain angle as a standard or unit, and then, by comparing the given angle with this, we obtain the magnitude of the former relative to that of the latter; just in the same manner as we calculate the magnitude of any given right line by comparing its length with that of one which is taken for a standard or linear unit, such as an inch, foot, or yard.



Thus, if we fix upon a right angle ACB, as the standard, or *angular unit*, and if MNO be the angle whose magnitude is required, we have only to draw CD making with CA the angle ACD equal to MNO *; and as ACD is to ACB as AD to AB (by ART. 168, GEOM.) the ratio of the arch AD to the *quadrant* (i. e. *fourth part* of the whole circle) AB, indicates the ratio of the angle ACD to a right angle. If we took *four* right angles, or the whole angular space round the point C, as the angular unit, then the ratio of the arch AD to the whole circumference ABEA, would indicate the ratio of the angle ACD to four right angles, by ART. 169, GEOM.

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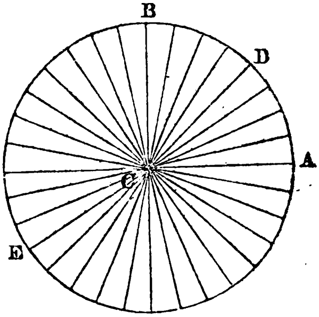
* Vide PROBLEM VIII. GEOMETRY.

Suppose, then, that we have taken four right angles as the angular unit; and likewise the radius CA of a given length, and that we have described the circle ABEA: we may compute the magnitude of MNO by the means aforesaid. For,

MNO : 4 right angles : : AD : ABEA.

Thus, if AD be one-tenth of ABEA, the angle MNO is one-tenth of 4 right angles, or of the whole angular space round C.

It is obvious that we should considerably facilitate the means of expressing the magnitudes of angles, by supposing the whole circumference ABEA divided into a number of small equal parts, as in the opposite figure, where that number is 32. *Ex. gr.* if the angle were ACD, then



ACD : 4 right angles : : 4 : 32 : : 1 : 8.

Therefore the magnitude of ACD would be expressed by saying that it was equal to 4 parts, or the 8th of 4 right angles. Instead of 32, however, the goniometrical circle is usually divided into 360 equal parts, and this number we will adopt in our Treatise.

Obs. 2. The goniometrical circle being supposed divided into 360 equal parts, each of these is called a *degree;* and any number of degrees is signified by placing ^{0} over the right shoulder of the number. Thus, *five* degrees is written 5^{0}; 360 degrees is written 360^{0}; and so in general.

A degree being supposed divided into *sixty* equal parts, each of these is called a *minute;* and any number of minutes is signified by placing an accent ′ over the right shoulder of the number. Thus, *seven* minutes is written $7'$; *sixty* minutes is written $60'$; and so in general.

A minute being supposed divided into *sixty* equal parts, each of these is called a *second;* and any number of seconds is signified by placing a double accent ″ over the right shoulder of the number. Thus, *nine* seconds is written 9″; *sixty* seconds is written 60″; and so in general *.

EXAMPLES.

75° 19′ 11″, means 75 degrees + 19 minutes + 11 seconds.
42° 10′ 58″, means 42 degrees + 10 minutes + 58 seconds.

Were the goniometrical circle divided, as above, into 360 equal parts, and right lines drawn from the centre to the several points of division, the whole angular space round C would be also divided into 360 equal parts, by ARTS. 85 and 86, GEOM. These parts might consequently be named *angular degrees;* and each angular degree would be divided into 60 *angular minutes*, each angular minute into 60 *angular seconds,* by the continued subdivision of the circumference as above. Hence, the angular parts corresponding exactly with the circumferential ones, the same notation would answer for both. Thus, 25° 9′ 4″ would signify an arch 25 degrees + 9 minutes + 4 seconds in circumferential length, or an angle 25 degrees + 9 minutes + 4 seconds in angular width. Any given number of degrees, minutes, or seconds, will, in like manner, express indifferently either an arch of the goniometrical circle, or the angle which it spans.

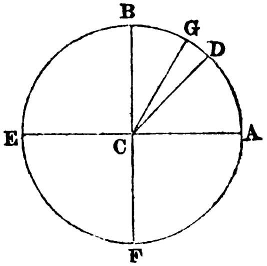
On the above principles,

360° will represent the whole circumference ABEA, *or* four right angles.

180° will represent the semicircumference ABE, *or* two right angles.

90° will represent the quadrant AB, *or* one right angle.

290° will represent three quadrants ABF, *or* three right angles.



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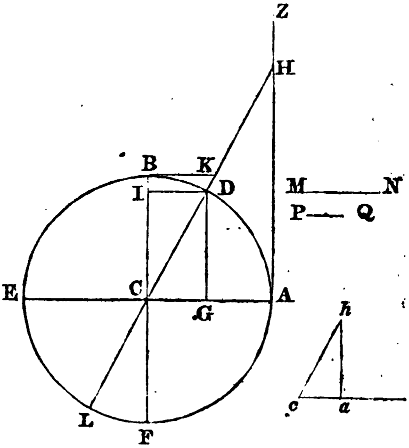
* Subdividing a second into sixty equal parts, each of these is called *third*, and is signified by three accents ‴; thus, 2‴ means two-thirds. a like sexagesimal division we get *fourths*, marked by four accents and thus proceed to fifths, marked by five accents ⁗′, &c.

45° will represent half a quadrant AD, *or* half a right angle.
60° will represent one-third of the semi-circumference AG, *or* two-thirds of a right angle. [Note A.]



Obs. 3. Instead of determining the magnitude of any given angle, such as ACD, by the ratio of its arch AD to the whole circumference, it is preferable to determine that magnitude by the ratio which some *right* line dependent upon the angle bears to the radius of the goniometrical circle, inasmuch as arches and circumferences do not enter into computation so easily as right lines. For instance, if GD, AH, be perpendicular to AC, meeting CD in D and H respectively, then the ratio of GD, or AH, to AC, determines the magnitude of the angle ACD. The reason is plain; because, by Art. 114, Geom., all right-angled triangles which have their sides about the right angles in the same ratio, are equiangular to each other: thus, if *ach* be any right-angled triangle, whose sides *ah, ac,* about the right angle have the same ratio to each other as AH to AC, the angle *ach* must be equal to the angle at ACH. Consequently, if in any angle *ach*, the ratio of the perpendicular *ah* to the line *ac* (between this perpendicular and the vertex of the opposite angle) be given, the angle *ach* is determinate, or fixed, and we may assign its magnitude by means of the goniometrical circle. *Ex. gr.* Suppose the ratio of *ah* to *ac* be given that of MN to PQ, we have only to raise AZ perpendicular to AC, and take AH in the same ratio to AC as MN to PQ; the angle ACH will be equal to the angle *ach* (Art. 114, Geom.), and therefore the magnitude of the latter is assigned by specifying the ratio of the former to the angular unit, or, in other words, the number of degrees, minutes, and seconds, it contains. Upon this principle rest the fundamental theorems of the present Science. We proceed to develop them in order.

LESSON II.

Fundamental Theorems, &c.

In the goniometrical circle, it is convenient to make a distinction between a tangent of the circle and a tangent of an *arch*; likewise between a secant of the circle and a secant of an *arch*. Thus, the indefinite right line, AZ, is a tangent of the circle, while only a portion of it, AH, is the tangent of the arch AD; the line, HL, is a secant of the same circle, while only a portion of it, HC, is the secant of the same arch AD. We speak also indifferently of the tangent, secant, &c. of the arch, and of the angle to which the arch corresponds. Hence,

Definition I. *The tangent of an arch, or angle,* is that portion of the geometrical tangent at one extremity of the arch, which is intercepted between the sides of the corresponding angle.

Thus, AH is the tangent of the arch AD, or of the angle ACD; and is generally written *tan* AD, or *tan* ACD [*].

Def. II. The *secant of an arch, or angle,* is that portion of a geometrical secant through one extremity of the arch, which is intercepted between a tangent at the other and the centre.

Thus, CH is the secant of the arch AD, or of the angle ACD; and is generally written *sec* AD, or *sec* ACD.

Article 1. *The square of the secant of an angle is equal to the sum of the squares of the tangent and of the radius.*

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[*] The *square* of *tan* ACD is thus written: tan^2 ACD; and in general the index is put, for brevity, over the right shoulder of the goniometrical *word*, instead of that of the whole quantity.

Let ACD be any angle represented by A. Then, if we represent AC by R, *sec*^{2} A = *tan*^{2} A + R^2.

DEMONSTRATION. By ART. 47, GEOM. CH^2 = AH^2 + AC^2. This was the assertion of the present Article.

DEF. III. The *complement* of an arch, or angle, is the difference between it and a quadrant, or right angle. Thus, in the last figure, the arch DB is the complement of the arch DA; and the angle DCB is the complement of the angle DCA.

Hence, if A represent any arch or angle, its complement will be represented by 90° — A.

Obs. 1. If BK be a tangent to the circle at B, then BK is the tangent of the arch BD, or of the angle BCD, *i. e.* BK is the tangent of the *complement* of AD, or of ACD. This tangent is written *tan. co.* ACD, meaning the tangent of the *complement* of ACD; or more usually it is written thus: *co-tan.* ACD, meaning the same thing. In the same manner CK is the secant of the complement of ACD, and is written *co-sec* ACD.

Obs. 2. If the complementary angle BCD be called B, it follows from this Article, that

$$sec^2\ B = tan^2\ B + R^2;$$

i. e. As the secant of B is the cosecant of A, and the tangent of B is the cotangent of A,

$$cosec^2\ A = cotan^2\ A + R^2.$$

ART. 2. *The rectangle under the tangent and cotangent of an arch, or angle, is equal to the square of the radius.*

Let ACD (fig. prec.) be any angle, represented by A. Then *tan* A × *cotan* A = R^2.

DEM. The angles HAC, ACB, CBK, being all right angles, AH is parallel to CB, and AC to BK. Consequently, by ART. 12. GEOM., the angle ACH = the angle CKB, and the angle AHC = the angle HCB. Therefore, by ART. 112, GEOM., AH : BC :: AC : BK, ∴ (ART. 100, GEOM.) AH × BK = BC × AC = AC^2. This, &c.

DEF. IV. *The sine of an arch, or angle,* is the perpendicular drawn from one extremity of the arch to the opposite side of the angle.

Thus DG (fig. prec.) is the sine of the arch AD, or of the angle ACD, and is usually written *sin* AD, or *sin* ACD.

Obs. If DI be perpendicular to CB, then DI is the sine of the arch BD, or of the angle BCD; *i. e.* DI is the sine of the complement of AD, or of ACD. This sine is usually written *co-sin* ACD, meaning, as before, the sine of the *complement* of ACD; it is yet more briefly written *cos* ACD.

As the triangles DGC, DIC, are equiangular to each other, and have the radius CD a common side, they have all their corresponding parts equal, by ART. 7, GEOM. Hence, GC is equal to DI, and therefore GC may be taken as the sine of BCD; *i. e.* as the cosine of ACD.

ART. 3. *The square of the radius is equal to the sum of the squares of the sine and cosine of an arch or angle.*

Let ACD (fig. prec.) be the angle represented by A. Then, $R^2 = \sin^2 A + \cos^2 A$.

DEM. By ART. 47, GEOM., $CD^2 = DG^2 + GC^2$. This, &c.

ART. 4. *The ratio of the sine to the cosine of an angle is the same as that of the tangent to the radius.*

Let ACD (fig. prec.) be the angle, represented by A, Then, $\frac{\sin A}{\cos A} = \frac{\tan A}{R}$.

DEM. By ART. 112, GEOM., DG : AH : : CG : AC; ∴ *alternando*, DG : CG : : AH : AC, (APPLICATION *of* ALGEBRA *to* GEOMETRY, *page* 19. 3°) $\frac{DG}{CG} = \frac{AH}{AC}$. This, &c.

Obs. Hence, if the complementary angle BCD be called B, it follows by this ART., that

$$\frac{\sin B}{\cos B} = \frac{\tan B}{R};$$

i. e. As the sine, cosine, and tangent of B are respectively the cosine, sine, and cotangent of A,

$$\frac{\cos A}{\sin A} = \frac{\text{cotan } A}{R}.$$

In other words, *the ratio of the cosine to the sine of an angle is the same as that of the cotangent to the radius.*

ART. 5. *The rectangle under the secant and cosine of an angle is equal to the square of the radius.*

Let ACD (fig. prec.) be the angle, represented by A. Then $sec\ A \times cos\ A = R^2$.

DEM. By Art. 112, GEOM., CH : CD :: CA : CG; ∴ by ART. 100, GEOM., $CH \times CG = CD \times CA = CA^2$. This, &c.

Obs. Hence it follows that, representing the angle BCD by B,

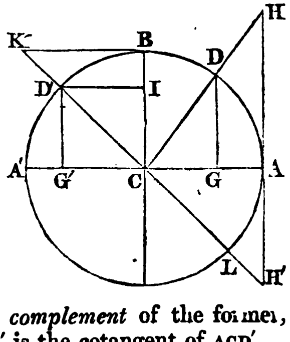
$$sec\ B \times cos\ B = R^2;$$

i. e. as the secant and cosine of B are respectively the cosecant and sine of A,

$$cosec\ A \times sin\ A = R^2.$$

DEF. V. *The versed sine of an arch or angle* is the intercept of the diameter between the sine and tangent. Thus GA (fig. prec.) is the versed sine of the arch AD, or of the angle ACD, and is written *ver. sin* ACD.

Obs. 1. It will easily be perceived that all the above definitions of goniometrical lines are applicable, whatever be the magnitude of the given angle, although we appear to have considered those *only*, such as ACD, which are less than a right angle.



Thus, if ACD' be the given angle, AH' will be its tangent, according to DEF. I.; because AH' is "that portion of the geometrical tangent intercepted between the two sides" CA, CD', (the latter being produced *backwards* through C, in order to meet the geometrical tangent at H'). Also BCD' being the difference between the angle ACD' and a right angle, is the *complement* of the former, by DEF. III., and therefore BK' is the cotangent of ACD'.

Again: CH' is the secant of ACD', according to DEF. II.; and CK', its cosecant.

Finally: D'G' is the sine of ACD', CG' its cosine, AG' its versed sine, according to DEF. IV. and *Obs.*, and DEF. V.

Obs. 2. Not only may the *definitions* given above be extended to the cases of such angles as ACD′, but likewise the *assertions* made in the preceding Articles and Observations.

Thus:

$CH'^2 = AH'^2 + AC^2$, that is, $sec^2\ ACD' = tan^2\ ACD' + R^2$. (Art. 1.)

$CK'^2 = BK'^2 + CB^2$, . . . $cosec^2\ ACD' = cotan^2\ ACD' + R^2$. (*Obs.* 2. Art. 1.)

$AH' : BC :: AC : BK'$, ... $tan\ ACD' \times cotan\ ACD' = R^2$ (Art. 2.)

$D'G'^2 + CG'^2 = R^2$, . . . $sin^2\ ACD' + cos^2\ ACD' = R^2$. (Art. 3.)

$D'G' : AH' :: CG' : AC$, . . . $\dfrac{sin\ ACD'}{cos\ ACD'} = \dfrac{tan\ ACD'}{R}$. (Art. 4.)

$D'I : BK' :: CI : CB$ . . . $\dfrac{cos\ ACD'}{sin\ ACD'} = \dfrac{cotan\ ACD'}{R}$. (*Obs.* Art. 4.)

$CH' : CD' :: CA : CG'$, . . . $sec\ ACD' \times cos\ ACD' = R^2$. (Art. 5.)

$CK' : CD' :: CB : CI$, . . . $cosec\ ACD' \times sin\ ACD' = R^2$. (*Obs.* Art. 5.)

Def. VI. The *supplement* of an arch or angle is the difference between it and a semicircle, or two right angles.

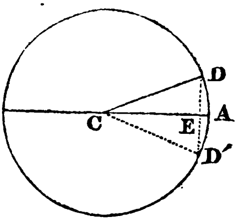
Thus, in the last figure, the arch D′A′ is the supplement of the arch D′A; and the angle D′CA′ is the supplement of the angle D′CA.

Hence, if A represent any arch or angle, its supplement will be represented by $180^\circ - A$. (Note B.)

Art. 6. *The sine of an angle is equal to the sine of its supplement.*

This is evident; for if ACD be any angle, then A′CD is its supplement, and these angles have both the same sine DG. This, &c.

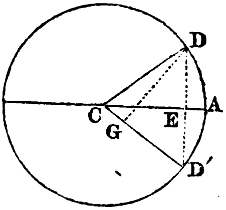
Art. 7. *The sine of an angle is equal to half the chord of twice the corresponding arch.*



Let ACD be any angle, and let the arch DAD′ be twice the arch DA, then the sine of ACD is equal to half the chord DD′.

DEM. Draw CD′. Inasmuch as the arch AD = the arch AD′, the angle ACD = the angle ACD′, by ARTS. 85 and 86, GEOM. Likewise in the triangles ACD, ACD′, the side CD = CD′, and the side CE is common. Therefore, by ART. 1, GEOM., the side DE = ED′. But as the angle CED = CED′, by ART. 2, GEOM., DE is perpendicular to CA, and .·. is the sine of ACD (DEF. IV.) : hence $sin\ ACD = \frac{DD'}{2}$. This, &c.

ART. 8. *The rectangle under the radius and the sine of twice an angle is equal to twice the rectangle under its sine and cosine.*



Let ACD be any angle represented by A; and let DCD′ = 2A. Then $R \times sin\ 2A = 2\ sin\ A \times cos\ A$.

DEM. Join D′D, and draw DG perpendicular to CD′. As in the preceding Article, CED′ is a right angle, and .·. = to DGD′. Likewise the angle DD′C is common to

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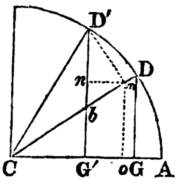
ART. 8^a. *The rectangle under the radius and sine of the sum of two angles is equal to the sum of the two rectangles under the sine of each and the cosine of the other.*



Let ACD, DCD′, be two angles, represented by A and B. Then $R \times sin\ (A + B) = sin\ A \times cos\ B + sin\ B \times cos\ A$.

DEM. Draw D′m perpendicular to CD, and mo, DG, D′G′, perpendicular to CA; likewise mn parallel to CA till it meets D′G′ in n. By ART. 112, GEOM. CD : Cm :: DG : mo, .·. by ART. 100, GEOM., CD × mo = Cm × DG; or, as mo is equal to the two triangles DD′G, ED′C; and ∴ these triangles are equiangular to each other. Consequently, by Art. 112, Geom., CD′ : DD′ :: CE : DG; and ∴, by Art. 100, Geom., CD′ × DG = DD′ × CE = 2DE × CE. But CD′ is the radius, DG is the sine of DCD′ or twice ACD, DE is the sine of ACD, and CE is the cosine of ACD; hence R × *sin* 2A = 2 *sin* A × *cos* A. This, &c.

nG′ (Art. 18, Geom.), ∴ CD × *n*G′ = DG × *cm*. Again, by Art. 11, Geom., the angles *cb*G′ and D′*bm* are equal, as also the right angles CG′*b* and D′*mb*; therefore the remaining angles *b*CG′, *b*D′*m*, are equal (Art. 39, Geom.). But the angle *mn*D′ being a right one, is equal, by Art. 8, Geom., to CGD; and consequently the triangles CGD, *n*D′*m*, are equiangular to each other. Wherefore, by Art. 112, Geom., CD : D′*m* :: CG : D*n*, ∴ CD × D′*n* = D′*m* × CG. By addition, ∴ CD × *n*G′ + CD × D′*n* = *cm* × DG + D′*m* × CG; that is, CD × (*n*G′ + D′*n*), or CD × D′G′ = DG × *cm* + D′*m* × CG. Now, CD is the radius, and D′G′ the sine of ACD′ or (ACD + DCD′); likewise DG is the sine, CG the cosine of ACD, and D′*m* is the sine, *cm* the cosine of DCD′. Hence, R × *sin* (A + B) = *sin* A × *cos* B + *sin* B × *cos* A. This, &c.

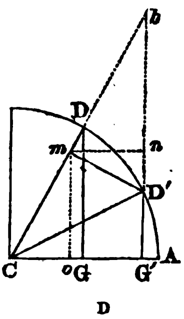
Obs. 1. The above is a general formula expressing the relation between the sine of the sum of *any* two angles and the sines and cosines of the angles themselves.

Obs. 2. If A = B, then the general formula becomes

R × *sin* 2 A = 2 *sin* A × *cos* A,

the same result as we obtained in Art. 8, by other means.

Art. 8^2. *The rectangle under the radius and sine of the difference of two angles is equal to the difference of the two rectangles under the sine of each and the cosine of the other.*



Let ACD, DCD′, be the angles, represented by A and B. Then R × *sin* (A − B) = *sin* A × *cos* B − *sin* B × *cos* A.

Dem. Draw, as before, D′*m* perpendicular to CD, and *mo*, DG, D′G′, perpendicular to CA; likewise *mn* parallel to CA till it meets D′G′ (produced so as to intersect CD in *b*). By Art. 112, Geom., CD : C*m* :: DG : *mo*, or *n*G′ (= *mo*, by Art. 20, Geom.) Therefore CD × *n*G′ = DG × C*m* (Art. 100, Geom.) Again, the right angles CG′*b*, D′*mb*, are equal, and the angle at *b* is common to the triangles

Art. 9. *The rectangle under the radius and the cosine of twice an angle is equal to the difference between the square of the radius and twice the square of the sine of the angle.*

Let ACD be any angle, represented by A; and let DCD′ $= 2A$. Then, $R \times \cos 2A = R^2 - 2\sin^2 A$.

Dem. By Art. prec.,

$$R \times \sin 2A = 2 \sin A \times \cos A.$$
$$\therefore R^2 \times \sin^2 2A = 4 \sin^2 A \times \cos^2 A;$$

but, by Art. 3, $\sin^2 2A = R^2 - \cos^2 2A$, and $\cos^2 A = R^2 - \sin^2 A$,

$$\therefore R^2 \times (R^2 - \cos^2 2A) = 4 \sin^2 A \times (R^2 - \sin^2 A),$$
$$\therefore R^4 - R^2 \times \cos^2 2A = 4 \sin^2 A \times (R^2 - \sin^2 A),$$
$$\begin{aligned}\therefore R^2 \times \cos^2 2A &= R^4 - 4\sin^2 A \times (R^2 - \sin^2 A),\\ &= R^4 - 4\sin^2 A \times R^2 + 4 \sin^4 A.\\ &= (R^2 - 2\sin^2 A)^2,\end{aligned}$$

$\therefore$ rooting both sides, $R \times \cos 2A = R^2 - 2 \sin^2 A$. This, &c.*

Obs. 1. If the angle A were divided into two equal parts, we should have, by the preceding formula,

$$R \times \cos 2\left(\frac{A}{2}\right) = R^2 - 2\sin^2\left(\frac{A}{2}\right);$$

$$i.\,e.\ \text{As } 2\left(\frac{A}{2}\right) = A,$$

$$R \times \cos A = R^2 - 2\sin^2 \frac{A}{2};$$

$CG'b$, $D'mb$; $\therefore$ these triangles are equiangular (Art. 39, Geom.), and $\therefore$ also the triangles CGD, $D'mb$. Consequently, by Art. 112, Geom., $CD : D'm :: CG : D'n$, $\therefore CD \times D'n = D'm \times CG$. By subtraction, $\therefore CD \times nG' - CD \times D'n = DG \times Cm - D'm \times CG$; that is, $CD \times (nG' - D'n)$, or $CD \times D'G' = DG \times Cm - D'm \times CG$. Now CD is the radius, $D'G'$ the sine of ACD′, (*i. e.* ACD − DCD′); likewise DG is the sine, CG the cosine of ACD, and $D'm$ is the sine, Cm the cosine of DCD′. Hence, $R \times \sin(A - B) = \sin A \times \cos B - \sin B \times \cos A$. This, &c.

* The theorem is easily derived from a geometrical consideration of the figure; but the above demonstration is preferable, first, as it is more *consecutive*, and, secondly, as exercising the student in goniometric calculation.

in other terms, *the rectangle under the radius and cosine of an angle is equal to the difference between the square of the radius and twice the square of the sine of half the angle;* which is in fact only the preceding theorem in another form of expression.

Obs. 2. By the preceding formula, inasmuch as

$\sin^2 \frac{A}{2} = R^2 - \cos^2 \frac{A}{2}$, (ART. 3), we have

$$R \times \cos A = R^2 - 2\left(R^2 - \cos^2 \frac{A}{2}\right)$$

$$= 2 \cos^2 \frac{A}{2} - R^2,$$

$$\therefore R^2 + R \times \cos A = 2 \cos^2 \frac{A}{2};$$

in other terms, *twice the square of the cosine of half an angle is equal to the square of the radius, together with the rectangle under the radius and the cosine of the angle.*

The preceding theorems are all which the reader will find necessary; let us collect them for the purpose of being easily gotten by heart, or consulted, into a table, which we shall call

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ART. 9^a. *The rectangle under the radius and the cosine of the sum of two angles is equal to the difference of the rectangles under the cosines and the sines of the angles.*

Let ACD, DCD′, be the angles, as in fig. ART. 8^a. Then, $R \times \cos(A + B) = \cos A \times \cos B - \sin A \times \sin B$.

DEM. By ART. 112, GEOM., $CD : Cm :: CG : CO$, $\therefore CD \times CO = CG \times Cm$. Again: as the triangles CGD, $nD'm$ are equiangular, $CD : D'm :: DG : nm$, (or $G'o$); $\therefore CD \times G'o = DG \times D'm$. By subtraction, $CD \times CO - CD \times G'o = CG \times Cm - DG \times D'm$; that is, $CD\,(CO - G'o)$, or $CD \times CG' = CG \times Cm - DG \times D'm$. Now CD is the radius, and CG′ the *cos* of ACD′, or (ACD + DCD′); likewise CG is the *cos* of ACD, Cm that of DCD′, and DG is the sine of ACD, $D'm$ that of DCD′. Hence $R \times \cos(A + B) = \cos A \times \cos B - \sin A \times \sin B$. This, &c.

Obs. If the two given angles be equal, the above formula becomes,

$$R \times \cos 2A = \cos A \times \cos B - \sin A \times \sin A$$
$$= \cos^2 A - \sin^2 A;$$

TABLE I.

1. $\sec^2 A = \tan^2 A + R^2.$
2. $\operatorname{cosec}^2 A = \cot an^2 A + R^2.$
3. $\tan A \times \operatorname{cotan} A = R^2.$
4. $\sin^2 A + \cos^2 A = R^2.$
5. $\dfrac{\sin A}{\cos A} = \dfrac{\tan A}{R}.$
6. $\dfrac{\cos A}{\sin A} = \dfrac{\operatorname{cotan} A}{R}.$
7. $\sec A \times \cos A = R^2.$
8. $\operatorname{cosec} A \times \sin A = R^2.$
9. $R \times \sin 2A = 2 \sin A \times \cos A.$
10. $R \times \cos 2A = R^2 - 2 \sin^2 A.$
11. $R \times \cos A = R^2 - 2 \sin^2 \dfrac{A}{2}.$
12. $R^2 + R \times \cos A = 2 \cos^2 \dfrac{A}{2}.$

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that is, *the rectangle under the radius and the cosine of twice an angle is equal to the difference between the square of the cosine and the square of the sine of that angle.*

ART. 93. *The rectangle under the radius and the cosine of the difference of two angles is equal to the sum of the rectangles under the cosines and the sines of the angles.*

Let ACD, DCD′, be the angles, as in fig. ART. 83. Then $R \times \cos (A - B) = \cos A \times \cos B + \sin A \times \sin B$.

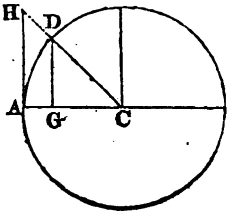
DEM. By ART. 112, GEOM., $CD : cm :: CG : co$, $\therefore CD \times co = CG \times cm$. Again, by ART. 14., GEOM., the angle CDG is equal to the angle cbG'; but the angle cbG', or mbD', is equal to the angle nmD', because in the triangles mbD', nmD', the angles $D'mb$, mnD', are both right, and the angle $mD'b$ is common; therefore the angles CDG and nmD' are equal. Consequently, as in the triangles CDG, nmD', these two angles are equal, as also the right angles CGD, mnD', the triangles are equiangular to each other (ART. 39, GEOM.) Wherefore, by ART. 112, GEOM., $CD : D'm :: DG : mn$, $\therefore CD \times mn = DG \times D'm$. By addition, therefore, $CD \times co + CD \times mn = CG \times cm + DG \times D'm$; that is, $CD \times (co + mn) = CG \times mc$

LESSON III.

Determination of remarkable Angles.

Let us now determine the magnitude of some particular angles from the ratio which their goniometrical lines have to the radius. It is by this means that we are able to compute the magnitude of all angles in general, as will briefly appear.

Art. 10. *The tangent of half a right angle is equal to the radius.*



Let ACD be $= 45^o$, *i. e.* half a right angle. Then, $tan\ 45^0 = R$.

Dem. In the triangle HAC the angle HAC is a right one, AH being a tangent of the circle at A; and HCA is given half a right angle. Consequently, by Art. 37, Geom., the angle AHC must be half a right angle. Hence, by Art. 4, Geom., the side AH is equal to the side AC; but AH is the *tan* of ACD, and AC is the radius. This, &c.

$+ DG \times D'm$; or, as $mn = CG'$, by Art. 20, Geom., we have $CD \times CG' = CG \times Cm + DG \times D'm$. Now CD is the radius, CG′ the cosine of ACD′ (*i. e.* ACD — DCD′); likewise CG, Cm, are the respective cosines of ACD, DCD′, and DG, $D'm$, are the respective sines of ACD, DCD′. Hence, $R \times cos\ (A - B) = cos\ A \times cos\ B + sin\ A \times sin\ B$. This, &c.

Art. 10ª. *The square of the secant of half a right angle is equal to twice the square of the radius.*

Dem. $sec^2\ 45^o = tan^2\ 45^o + R^2$, by Art. 1;
$= R^2 + R^2$, by Art. 10;
$= 2R^2$.

This, &c.

Art. 10ᵇ. *The cotangent of half a right angle is equal to the radius.*

Dem. $tan\ 45^o \times cotan\ 45^o = R^2$, by Art. 2;
$\therefore\ R \times cotan\ 45^o = R^2$, by Art. 10;
$\therefore\ cotan\ 45^o = R$.

This, &c.

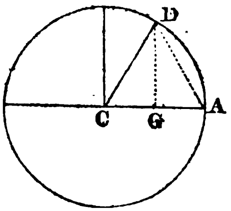
Art. 11. *The sine is equal to the cosine of half a right angle, and the square of each is equal to half the square of the radius.*

In the fig. above, let ACD be $= 45^\circ$. Then, $sin\ 45^\circ = cos\ 45^\circ$, and $sin^2\ 45^\circ = cos^2\ 45^\circ = \frac{R^2}{2}$.

Dem. Draw DG perpendicular to AC; and for the same reasons as above, $DG = CG$. Likewise, Art. 47, Geom., $CD^2 = DG^2 + CG^2$; that is, $CD^2 = 2DG^2$. Hence, $DG^2 = CG^2 = \frac{CD^2}{2}$. But DG is the sine, and CG the cosine of ACD, and CD is the radius. This, &c.

Art. 12. *The cosine of an angle equal to one-third of two right angles is equal to half the radius.*

Let ACD be $= 60^\circ$, *i. e.* one-third of two right angles. Then, $cos\ 60^\circ = \frac{R}{2}$.



Dem. Join AD. The three angles of the triangle ACD are equal to two right angles, by Art. 37, Geom.; but the angle ACD is given equal to *one*-third of two right angles, therefore the remaining angles ADC and DAC must be together equal to *two*-thirds of two right angles. Wherefore, as $CD = CA$, the angle $ADC =$ the angle DAC, and each of them is equal to *one*-third

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Art. 10*. *The square of the cosecant of half a right angle is equal to twice the square of the radius.*

Dem. $cosec\ 45^\circ = cotan^2\ 45^\circ + R^2$, by *Obs.* 2, Art. 1;
$= R^2 + R^2$, by Art. prec.;
$= 2R^2$.

This, &c.

Obs. These three results might have been obtained with great ease geometrically, as will be evident by drawing a proper figure.

of two right angles. Consequently, by Art. 122, Geom. the three sides AC, CD, DA, are equal; and therefore, by Art. 139, Geom., the side CA is equally divided at the point G, by the perpendicular DG, i. e. $CG = \frac{CA}{2}$. But CA is the radius, and CG the cosine of ACD. This, &c.

Art. 13. *The square of the sine of an angle equal to one-third of two right angles is equal to three-fourths of the square of the radius.*

In the figure above, let $ACD = 60^\circ$. Then $sin.^2\, 60^\circ = \frac{3R^2}{4}$.

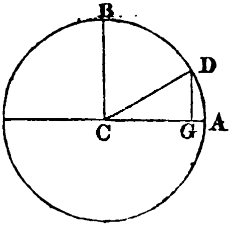
Dem. $sin^2\, 60^\circ + cos^2\, 60^\circ = R^2$, by Art. 3;

$\therefore\ sin^2\, 60^\circ + \frac{R^2}{4} = R^2$, by Art. prec.;

$\therefore\ sin^2\, 60^\circ = R^2 - \frac{R^2}{4} = \frac{3R^2}{4}$.

This, &c.

Art. 14. *The sine of an angle equal to one-third of a right angle is equal to half the radius.*



Let ACD be $= 30^\circ$, *i. e.* one-third of a right angle. Then, $sin\ 30^\circ = \frac{R}{2}$.

Dem. Since the angle ACD is one-third of a right angle, the angle BCD is two-thirds of a right angle; that is, BCD is one-third of two right angles. Consequently, by Art. 12, $cos\ BCD = \frac{R}{2}$; but, Def. IV. *Obs.*, the cosine of an angle is the sine of the complement of that angle, and therefore $cos\ BCD = sin\ ACD$. Hence, $sin\ ACD = \frac{R}{2}$. This, &c.

Art. 15. *The square of the cosine of an angle equal to one-third of a right angle is equal to three-fourths of the square of the radius.*

In the figure above, let ACD be $= 30^\circ$. Then $cos^2 30^\circ = \frac{3R^2}{4}$.

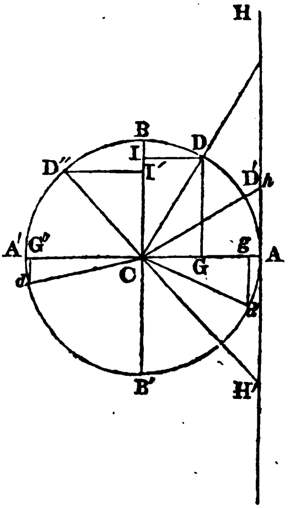
$$\text{Dem. } sin^2 30^\circ + cos^2 30^\circ = R^2, \text{ by Art. 3;}$$

$$\therefore \frac{R^2}{4} + cos^2 30^\circ = R^2, \text{ by Art. prec.;}$$

$$\therefore cos^2 30^\circ = R^2 - \frac{R^2}{4} = \frac{3R^2}{4}. \text{ This, \&c.}$$

Obs. 1. If we consider only such angles as are less than a right angle, it is plain from the Articles in this Lesson, that we can determine the magnitude of an angle in some cases, by knowing certain of its goniometrical lines. Thus, if we know that the tangent of an angle is equal to the radius, or that its sine, or cosine, is to the radius as $1 : \sqrt{2}$, the angle must necessarily be equal to half a right angle, by ARTS. 10 and 11, respectively. If we know that the cosine of an angle is equal to half the radius, or that its sine is to the radius as $\sqrt{3} : 2$, the angle must necessarily be equal to one-third of two right angles, by ARTS. 12 and 13, respectively. If we know that the sine of an angle is equal to half the radius, or that its cosine is to the radius as $\sqrt{3} : 2$, the angle must necessarily be equal to one-third of a right angle, by ARTS. 14 and 15 respectively.

Obs. 2. There are likewise certain other angles immediately determinable from their goniometrical lines, which we proceed to enumerate. It appears, from ART. 6, that the magnitude of an angle is not generally determinable from the ratio of its *sine* to the radius, inasmuch as the same sine corresponds to *two* angles, viz. those which are *supplementary* one to the other. But in order to prevent the same ambiguity with regard to the other goniometrical lines, mathematicians have recourse to the following expedient: All perpendiculars to the diameter AA′, such as DG, BC, at the *upper* side of AA′, are affected with the sign +; and those, such as AH′, CB′, at the *under* side of AA′, are affected with the sign —. Likewise: All perpendiculars to the diameter BB′, such as DI, CG, at the right side of BB′, are affected with the sign +; and those, such as D″I′, CA′, at the *left* side of BB′, are affected with the sign —. Thus, AH′ is written — AH′, which distinguishes it from + AH, to which it may be equal in length; and D″I′ is written — D″I′, which distinguishes it from + DI, to which it may be equal in length. It is, however, only necessary to write the *minus* signs to their corresponding lines, the others being always understood to have the sign +.



Now, if the radius CD be supposed to revolve round the point C, so as to form with its first position CA, angles of increasing magnitude, it is plain that the sine DG beginning from nothing (when CD coincides with CA), will continually augment, as at D, till CD reaches the point B, where it will be equal to CB the radius; thenceforward it will continually diminish till it vanishes at the point A′ into nothing. Hence, at the three remarkable points, A, B, A′, the magnitude of the sine will be thus expressed algebraically:

Sin $0^\circ = 0$, meaning that 0 is the limit of size from which the sine begins, the revolving radius setting out from A.

Sin $90^\circ = R$.

Sin $180^\circ = 0$, meaning that 0 is the limit to which the sine approaches, the revolving radius coming to A′.

Again: on the same supposition, it appears that the cosine CG, beginning from equality with the radius (when CD coincides with CA), will continually diminish, as at D, till CD reaches B, where it will vanish; thenceforward it will continually augment till it again becomes equal to the radius, when CD coincides with CA′. Hence, at the same remarkable points A, B, A′, the magnitude of the cosine will be thus expressed algebraically:

Cos $0^\circ = R$, meaning that the radius is the limit from which the cosine begins, the revolving radius setting out from A.

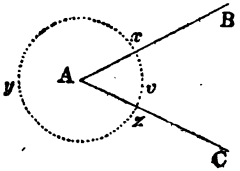
Cos $90^{\circ} = 0$, meaning that 0 is the limit to which the cosine approaches, the revolving radius coming to B.

Cos $180 = -$ R, meaning that the *left-hand* radius is the limit to which the cosine approaches, the revolving radius coming to A′.

Finally: On the same supposition it appears that the tangent A*h* beginning from nothing (when CD coincides with CA), will continually augment, as at D′, and become at length indefinitely great, when CD is at the point B; for *then* the geometrical tangent AH being parallel to one side, CB, of the angle ACB, the portion of it intercepted between both sides will be immeasurably great. When the revolving radius gets beyond B, as at D″, then the intercepted portion abovementioned will be AH′, contained between the sides CA and CD″, the latter being produced *backwards* through C, so as to meet the geometrical tangent at H′. When the radius comes to A′, the tangent again vanishes. Hence, at the same remarkable points A, B, A′, the magnitude of the tangent will be thus expressed algebraically (the sign ∞ being generally used to designate a quantity indefinitely great):

Tan $0^{\circ} = 0$, meaning that 0 is the limit from which the tangent begins, the revolving radius setting out from A.

Tan $90^{\circ} = \infty$, meaning that a quantity indefinitely great is that to which the tangent approaches, the revolving radius coming to B.



Obs. 2^2. The definition of an angle (DEF. I, GEOM.) applies to the space from *x* round through *y* to *z*, as well as to that from *x* through *v* to *z*. Hence, in the above figure, the revolving radius may come into the position *cd*, and thus form an angle greater than two right angles. The sine of such an angle, viz. *d*G″, will have the sign —, and the cosine, CG″, likewise; according to what we have said in the preceding part of *Obs.* 2. When the radius comes to B′ the sine will be

Tan 180° = 0, meaning that 0 is the limit to which the tangent approaches, the revolving radius coming to A′.

The preceding formulæ may be registered in a second table, which will be found useful.

TABLE II.

$\tan 45^\circ = \text{R}$	$\sin 0^\circ = 0$
$\sin 45^\circ = \dfrac{\text{R}}{\sqrt{2}}$	$\cos 0^\circ = \text{R}$
$\cos 45^\circ = \dfrac{\text{R}}{\sqrt{2}}$	$\tan 0^\circ = 0$
$\cos 60^\circ = \dfrac{\text{R}}{2}$	$\sin 90^\circ = \text{R}$
$\sin 60^\circ = \dfrac{\sqrt{3}}{2}.\text{R}$	$\cos 90^\circ = 0$
$\sin 30^\circ = \dfrac{\text{R}}{2}$	$\tan 90^\circ = \infty$
$\cos 30^\circ = \dfrac{\sqrt{3}}{2}.\text{R}$	$\sin 180^\circ = 0$
	$\cos 180^\circ = -\text{R}$
	$\tan 180^\circ = 0$

From the above table we have a ready means of determining the magnitude of such angles as have any of the sines, cosines, or tangents, there registered. For example,

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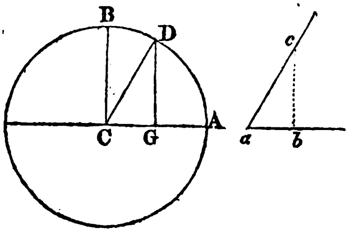
greatest, and will then = — R, while the cosine at that point will vanish. After this the sine *d′g* will diminish, but still have the sign —, while the cosine c*g* will augment, and have the sign +. At A the sine will vanish, and the cosine become = R.

In the higher branches of Mathematics it is convenient to introduce such angles, and even to proceed farther; but in our series of elementary treatises it will be only necessary to consider angles between 0° and 180°. From what we have said, however, it is manifest that

$$\sin 290^\circ = -\text{R}\,;\ \cos 290^\circ = 0,$$
$$\sin 360^\circ = 0\,;\ \cos 360^\circ = \text{R}.$$



if it be required to find the magnitude of the acute angle *bac*, upon being given that the intercept *ab*, between the perpendicular from one of its sides on the other, is *half* the distance *ca*; this angle is plainly equal to one-third of two right angles. Because, taking in the goniometric circle, a cosine CG equal half the radius, the triangle GCD will be equiangular to *bac* (ART. 115, GEOM), inasmuch as the angles at G and *b* are equal, and CG : CD : : *ba* : *ac*. Consequently the angle *bac* = the angle GCD; but the angle GCD is = 60°, because its cosine $= \frac{R}{2}$, and therefore also the angle *bac* = 60°.

Table II. would suffice to determine the magnitude of but very few angles. It therefore becomes an object to construct another in which the sines, cosines, &c., of many different angles shall be registered; so that whatever be the ratio of the three lines *ab*, *ac*, *cb*, amongst each other, we may find it in the Table opposite some angle, which will therefore be equal to the angle *bac*, whose magnitude is thereby determined. Such a Table has been constructed, but its insertion here would be neither suitable for our treatise, nor of any use to our readers. We shall merely explain its principles.

If in the formula 11, TABLE I, we take A = 30°, then by TAB. II. $cos\ A = \frac{R \cdot \sqrt{3}}{2}$, and the formula becomes

$$R \times \frac{R\sqrt{3}}{2} = R^2 - 2\,sin^2\,15^\circ,$$

$$\therefore\ 2\,sin^2\,15^\circ = R^2 - R^2\,\frac{\sqrt{3}}{2},$$

$$\therefore \quad sin\ 15^\circ = \sqrt{\frac{R^2}{2} - \frac{R^2\sqrt{3}}{4}},$$

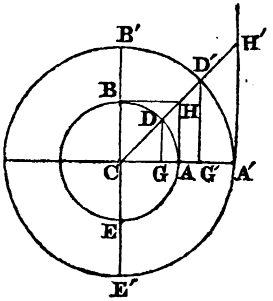
$$= R\sqrt{\tfrac{1}{2} - \frac{\sqrt{3}}{4}}.$$

We have thus found the value of the sine of an angle = 15°; and by help of the same formula, we may find, in the same manner, the value of the sines of angles equal respectively to 7° 30′, 3° 45′, &c., &c., each successive angle being *half* the preceding. Likewise, inasmuch as the sine of an angle is equal to the cosine of its complement, the values thus found for the *sines* of angles equal respectively to 15°, 7° 30′, 3° 45′, &c. &c., will be the values for the *cosines* of angles equal respectively to 75°, 82° 30′, 86° 15′, &c. &c., which are the complements of the former. We might therefore register these two series of sines and cosines, which would enable us, in like manner as TABLE II, to determine the magnitude of several angles. By a similar process we may compute the values of the sines, cosines, tangents, &c., of innumerable angles, and these values being registered, serve likewise to the above purpose. This is the principle on which the required Table is constructed, though by the help of more powerful formulæ than the above, which is, however, sufficient to afford an explanation of the system.

LESSON IV.

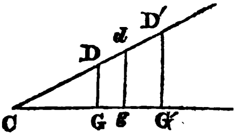
Numeral Sine, Cosine, &c.

Obs. 1. We have hitherto considered the different goniometric lines as referred to the radius of a particular circle ABE (fig. p. 39); but it is evident that the exact length of CA is of no importance to the conclusions derived in the preceding Lessons: these would have followed, though we had chosen CA half, or twice, the length it appears, or of any length whatsoever. In determining the magnitude of angles, we are therefore at perfect liberty to assume any circle we please as the goniometric one, and to calculate sines, cosines,



&c., in relation to its radius. We have only to recollect, that if we choose ABE as the goniometric circle, DG, CG, AH, are respectively the sine, cosine, and tangent, of the angle ACD, referred to the radius CA; while, if we choose any other circle A′B′E′ as the goniometric one, D′G′, CG′, A′H′, are respectively the sine, cosine, and tangent, of that angle referred to the new radius CA′.

It thus appears that the sine, cosine, &c. of an angle are not any particular lines of a certain magnitude, but lines of a magnitude proportional to the radius of the goniometric circle to which they respectively belong. However, although these goniometric lines will vary in length, according to the variation of the radius to which they are referred, yet their *ratio* to their respective radii will not. Thus, if ABE, A′B′E′, be two goniometric circles, DG will be the sine of the angle ACD referred to CD as radius, as D′G′ will be the sine of the same angle A′CD′ referred to CD′ as radius; but, by ART. 112, GEOM. DG : DG′ : : CD : CD′ ∴ *alternando* DG : CD : : DG′ : CD′, i. e. the ratio of the smaller sine to its corresponding radius is the same as the ratio of the greater sine to its corresponding radius. In like manner CG : CD : : CG′ : CD′; that is, the cosines of the *same* angle in *different* circles are to their corresponding radii in the same ratio; and AH : CA : : A′H′ : CA′, that is, the tangents of the same angle in different circles are to their corresponding radii in the same ratio. From hence it follows, that the magnitude of an angle is as well determined by one goniometric circle as another; for if, in the angle *bac* (fig. p. 60), the line *cb* be perpendicular, whatever be the given ratio between any two of the three sides *ca, ab, bc,* let us take any corresponding two of the goniometric lines, DC, CG, GD, and also any corresponding two of the goniometric lines, D′C, CG′, G′D′, in the given ratio, and as the triangles *bac,* GCD, G′CD′, are all right angled, the angles *bac,* GCD, G′CD′, will be all equal, by ART. 115, GEOM. Consequently the angle *bac* will come out to be of the same magnitude, whether it be computed in the one circle or in the other.



Obs. 2. Inasmuch as, whatever length we take the radius, CD, or CD′, we shall have CD : DG : : CD′ : D′G′, from hence it follows, by DEF. XXXV, GEOM., that any submultiple of CD is as often contained in DG as an equi-submultiple of CD′ is contained in D′G′. Wherefore, if DG be supposed to consist of a certain number of equal parts into which CD may be divided, D′G′ will consist of the same number of equi-submultiples of CD′. For example, if CD be divided into five equal parts, and if DG be equal to *three* of these, then D′G′ must also be equal to three fifth parts of CD′. Consequently, when we wish to express the sine of an angle, instead of declaring what length of radius we use, it is sufficient if we declare in numbers what *part* of our radius the sine is: because, *any* right line being assumed as radius, it is only necessary to take the same part *of it*, and this will be the sine of the same angle, only referred to a new radius. Thus, if we declare the sine of an angle to be *one-half* of *our* radius, it immediately points out to others an angle of 30°; because, *whatever* radius be assumed, if one-half of it be taken for a sine, the corresponding angle will be the same in magnitude, viz. (ART. 14.) an angle = 30°.

The above considerations give rise to the system of *numeral* sines, cosines, &c, instead of those heretofore used, which are *linear*. For:

If we represent by the number y the right line dg, from any point in one side of an angle perpendicular on the other, and by the number d the distance cd, from that point to the vertex, then we shall have

$$\text{DG} : \text{CD} :: y : d,$$

i. e. representing the angle at C by A,

$$\mathit{sin}\ \text{A} : \text{CD} :: y : d,$$

or, *whatever* length (CD or CD′) we take for radius R,

$$\mathit{sin}\ \text{A} : \text{R} :: y : d \ \therefore\ \mathit{sin}\ \text{A} = \frac{y}{d} \cdot \text{R};$$

consequently the number $\frac{y}{d}$ will express what part of any assumed radius (CD or CD′) the corresponding sine is. Thus, if $dg : cd :: 3 : 5$, then

$$sin\ \text{A} : \text{R} :: 3 : 5 \ \therefore\ sin\ \text{A} = \tfrac{3}{5}.\text{R};$$

that is, the sine of the angle is three-fifths of its corresponding radius.

Now if we consider the radius R, whatever be its length, as a *linear unit* (AP. of ALG. to GEOM. p. 16), and *sin* A as made up a certain number of its submultiple parts, — the *number* 1 may always stand for this radius, and then the *number* $\frac{y}{d}$ will represent *sin* A; because, by the above proportion $sin\ \text{A} : \text{R} :: \frac{y}{d} : 1$.

Unity, or 1, being taken to represent the linear radius, R, is denominated (though with no slight degree of impropriety) the *numeral* radius. Conformably with this now universally adopted title, we may call the number $\frac{y}{d}$ (i. e. the number which denotes the ratio of the perpendicular above described to the corresponding distance) the *numeral sine;* because it is to unity, the numeral radius, as the linear sine is to the linear radius. Hence, in algebraical language, we may write

$$sin\ \text{A} = \frac{y}{d},$$

carefully distinguishing *this sin* A from the *linear sin* A, by the recollection that when we meet *sin* A put equal to a *number*, the sine itself is considered as a number, referred to 1, as the linear *sin* A is referred to its corresponding radius. Thus, when we see the formula, $sin\ \text{A} = \frac{\sqrt{3}}{2}$, this only means, that the *numeral* sine of the angle represented by A is $\frac{\sqrt{3}}{2}$, — being to 1 as the linear sine of the same angle is to R, its corresponding radius.

In like manner, if we represent by the number x, the intercept $\mathrm{C}g$ between y and the vertex, we shall have

$$\mathrm{CG} : \mathrm{CD} :: \cos \mathrm{A} : \mathrm{R} :: x : d :: \frac{x}{d} : 1;$$

the number $\frac{x}{d}$ will therefore represent the linear cosine of the angle, and being itself called the *numeral cosine*, we may write

$$\cos \mathrm{A} = \frac{x}{d}.$$

This is to be distinguished as above from the linear $\cos \mathrm{A}$, which is $= \frac{x}{d} \cdot \mathrm{R}$.

$$\text{Finally, } \mathrm{DG} : \mathrm{CG} :: \tan \mathrm{A} : \mathrm{R} :: y : x :: \frac{y}{x} : 1,$$

$$\text{and, } \mathrm{CD} : \mathrm{CG} :: \sec \mathrm{A} : \mathrm{R} :: d : x :: \frac{d}{x} : 1,$$

$$\text{and, } \mathrm{CG} : \mathrm{DG} :: \mathrm{cotan}\, \mathrm{A} : \mathrm{R} :: x : y :: \frac{x}{y} : 1,$$

$$\text{and, } \mathrm{CD} : \mathrm{DG} :: \mathrm{cosec}\, \mathrm{A} : \mathrm{R} :: d : y :: \frac{d}{y} : 1;$$

hence we may write *numeral* tangent, secant, cotangent, and cosecant of an angle as follows:

$$\tan \mathrm{A} = \frac{y}{x},$$

$$\sec \mathrm{A} = \frac{d}{x},$$

$$\mathrm{cotan}\, \mathrm{A} = \frac{x}{y},$$

$$\mathrm{cosec}\, \mathrm{A} = \frac{d}{y}.$$

Both the linear and numeral values for *sin, cos,* &c., are used, perhaps too indiscriminately, by scientifical writers; it was therefore necessary to propound both systems; but we shall find it most convenient hereafter to use the numeral values chiefly, instead of the linear, as we have done till now. Computation will be much facilitated by this means; and the results obtained will be far more general than on the other system, inasmuch as by taking unity for radius, this may be considered to stand for *any* particular linear radius which can be assumed, and therefore for all.

From the preceding formulæ we will now draw up a table analogous to TABLE I, the several terms *sin, cos,* &c., now being considered as numbers, and the *radius* as 1.

ART. 16. 1°. $sec^2 \text{A} = \frac{d^2}{x^2} = \frac{y^2 + x^2}{x^2} = \frac{y^2}{x^2} + 1 = tan^2 \text{A} + 1.$

2°. $cosec^2 \text{A} = \frac{d^2}{y^2} = \frac{y^2 + x^2}{y^2} = 1 + \frac{x^2}{y^2} = 1 + cotan^2 \text{A}.$

3°. $tan \text{A} \times cotan \text{A} = \frac{y}{x} \times \frac{x}{y} = 1.$

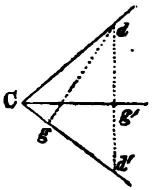
4°. $sin^2 \text{A} + cos^2 \text{A} = \frac{y^2}{d^2} + \frac{x^2}{d^2} = \frac{y^2 + x^2}{d^2} = \frac{d^2}{d^2} = 1.$

5°. $\frac{sin \text{A}}{cos \text{A}} = \frac{y}{d} \div \frac{x}{d} = \frac{y}{x} = tan \text{A}.$

6°. $\frac{cos \text{A}}{sin \text{A}} = \frac{x}{d} \div \frac{y}{d} = \frac{x}{y} = cotan \text{A}.$

7°. $sec \text{A} \times cos \text{A} = \frac{d}{x} \times \frac{x}{d} = 1.$

8°. $cosec \text{A} \times sin \text{A} = \frac{d}{y} \times \frac{y}{d} = 1.$



Also, let the angle dcg' be A, and the angle dcd' be = twice A. Draw dg perpendicular to cg, and taking $cd' = cd$, join dd'. Then, by **Art.** 140, **Geom.**, $dg' = g'd'$, and dg' is perpendicular to cg'. Hence, inasmuch as the angles $d'gd$, $d'g'c$, are equal, being both right, and the angle at d' is common to the two triangles $d'gd, d'g'c$, these triangles are equiangular to each other. Consequently, by **Art.** 112, **Geom.**,

$$dg : cg' : : dd' : cd' ;$$

or, representing (as before) dg by y, cg' by x', dg' by y', and cd' or cd by d,

$$y : x' : : 2\,y' : d,$$

$$\therefore\ y = 2\,y' . \frac{x}{d},$$

$$\therefore\ \frac{y}{d} = \frac{2\,y'}{d} . \frac{x'}{d} .$$

But, according to *Obs.* 2, Lesson IV,

$$\sin 2\text{A} = \frac{y}{d}, \sin \text{A} = \frac{y'}{d}, \cos \text{A} = \frac{x'}{d};$$

$$\therefore\ 9^\circ,\ \sin 2\text{A} = 2 \sin \text{A} . \cos \text{A}.$$

Also, by 9°, $\sin^2 2\text{A} = 4 \sin^2 \text{A} . \cos^2 \text{A}$;
but, by 4°, $\sin^2 2\text{A} + \cos^2 2\text{A} = 1$, and $\cos^2 \text{A} = 1 - \sin^2 \text{A}$,

$$\begin{aligned}\therefore\ 1 - \cos^2 2\text{A} &= 4 \sin^2 \text{A} . (1 - \sin^2 \text{A}),\\ &= 4 \sin^2 \text{A} - 4 \sin^4 \text{A},\\ \therefore\ \cos^2 2\text{A} &= 1 - 4 \sin^2 \text{A} + 4 \sin^4 \text{A},\\ &= (1 - 2 \sin^2 \text{A})^2,\\ \therefore\ 10^\circ, \cos 2\text{A} &= 1 - 2 \sin^2 \text{A}\ *.\end{aligned}$$

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* Combining 9° and 10°, we may obtain a formula, expressing the

Also, by 10°, $cos\ 2\left(\frac{A}{2}\right) = 1 - 2\,sin^2\frac{A}{2}$,

i. e. as $2\left(\frac{A}{2}\right) = A$,

$$11°,\ cos\ A = 1 - 2\ sin^2\frac{A}{2}.$$

Finally, by 11°, $1 + cos\ A = 2 - 2\ sin^2\frac{A}{2}$.

$$= 2\left(1 - sin^2\frac{A}{2}\right),$$

$$= 2\ cos^2\frac{A}{2},\ \text{by } 4°,$$

$$\therefore\ 12°,\ 1 + cos\ A = 2\ cos^2\frac{A}{2}.$$

Let us register these conclusions in

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relation between the *tangent* of an angle and the *tangent* of its double, thus:

By 5°, $tan\ 2A = \frac{sin\ 2A}{cos\ 2A} = \frac{2\ sin\ A \times cos\ A}{1 - 2\ sin\ A} = \frac{2 \cdot \frac{sin\ A}{cos\ A}}{\frac{1}{cos^2 A} - 2 \cdot \frac{sin^2 A}{cos^2 A}}$

(dividing both numerator and denominator by *cos* A).

But by 5°, $\frac{sin\ A}{cos\ A} = tan\ A$, and $\frac{sin^2 A}{cos^2 A} = tan\ A$; likewise, by 7°, $\frac{1}{cos^2 A} = sec^2 A$. Hence, $tan\ 2A = \frac{2\ tan\ A}{sec^2 A - 2\ tan^2 A}$; or, as $sec^2 A = tan^2 A + 1$, (by 1°), we have

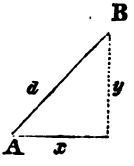
$$tan\ 2A = \frac{2\ tan\ A}{1 - tan^2 A}.$$

TABLE III.

1. $\sec^2 A = \tan^2 A + 1.$
2. $\operatorname{cosec}^2 A = \cot an^2 A + 1.$
3. $\tan A \,.\, \operatorname{cotan} A = 1.$
4. $\sin^2 A + \cos^2 A = 1.$
5. $\dfrac{\sin A}{\cos A} = \tan A.$
6. $\dfrac{\cos A}{\sin A} = \operatorname{cotan} A.$
7. $\sec A \,.\, \cos A = 1.$
8. $\operatorname{cosec} A \,.\, \sin A = 1.$
9. $\sin 2A = 2 \sin A \,.\, \cos A.$
10. $\cos 2A = 1 - 2 \sin^2 A.$
11. $\cos A = 1 - 2 \sin^2 \dfrac{A}{2}.$
12. $1 + \cos A = 2 \cos^2 \dfrac{A}{2}.$

By comparing this table with TABLE I. it is plain that there is no difference whatever, except that in this the quantities are numerical, in that geometrical; *unity*, the numeral radius, here taking the place of R, the linear radius. We might, indeed, have derived this table at once from the other, merely by substituting 1 for R, and then considering the *sin*, *cos*, &c., as numbers; but such a process might have tended to confound the two systems, linear and numeral, in the reader's mind, while it is of great importance that he should keep them distinct. In fact, we should have preferred giving the numeral system alone, but that the other, besides exhibiting in a natural way the origin of the science, is the one mostly used in scientifical works, especially British.

ART. 17. If the angle A be half a right angle, or 45°, then also the angle B, formed by the perpendicular y, from a point in one of its sides on the other, is half a right angle; and $\therefore$ the side y = the side x. Hence,



$$\cos 180^\circ = \frac{d}{x} = -1,$$ According to *Obs.* 2, Art. 15.

$$\tan 180^\circ = \frac{0}{x} = 0.$$

The results obtained in this Observation may be collected together for reference in

TABLE IV.

$\tan 45^\circ$	=	1	$\sin 0^\circ$	=	0
$\sin 45^\circ$	=	$\frac{1}{\sqrt{2}}$	$\cos 0^\circ$	=	1
$\cos 45^\circ$	=	$\frac{1}{\sqrt{2}}$	$\tan 0^\circ$	=	0
$\cos 60^\circ$	=	$\frac{1}{2}$	$\sin 90^\circ$	=	1
$\sin 60^\circ$	=	$\frac{\sqrt{3}}{2}$	$\cos 90^\circ$	=	0
$\sin 30^\circ$	=	$\frac{1}{2}$	$\tan 90^\circ$	=	∞
$\cos 30^\circ$	=	$\frac{\sqrt{3}}{2}$	$\sin 180^\circ$	=	0
			$\cos 180^\circ$	=	-1
			$\tan 180^\circ$	=	0

This Table is to be made use of in the same way as Table II. for determining the magnitude of angles, and also for constructing another, in which the sines, cosines, &c. of several angles should be registered, in order to determine the magnitude of such angles when we happen to know that of their sine, cosine, &c.

We have thus deduced the elements of goniometry on the numerical as well as the geometrical system, in a separate and independent manner. This has undoubtedly lengthened our Treatise, but we hope it will prevent that confusion which is apt to arise in the mind of a reader upon finding the two systems used indiscriminately, as they are in most works on the science.

Obs. 1. As we have only to blot out the R in TABLES I. and II. in order to get the formulæ of TABLES III. and IV. so, reversely, if we would obtain a geometrical formula corresponding to any numerical formula in this science, we have only to introduce the linear radius, R, into the latter, according to TABLES I. and II. Thus, if we have the numeral formulæ,

$$sec^2\ A = tan^2\ A + 1, \quad \frac{sin\ A}{cos\ A} = tan\ A, \quad cos\ 60^\circ = \frac{1}{2}.$$

In order to obtain the corresponding geometrical formulæ, we have only to substitute the geometrical radius, R, for the numeral radius 1, as follows:

$$sec^2\ A = tan^2\ A + R^2, \quad \frac{sec\ A}{cos\ A} = \frac{tan\ A}{R}, \quad cos\ 60^\circ = \frac{R}{2}.$$

The geometrical formulæ are immediately discoverable from the numerical, without referring to the Tables, if we are always careful to introduce the radius R, and its powers, so that the resulting formula may have *all* its terms of the same species of quantity, *i. e.* all lines, or all surfaces, &c. Thus having the formula,

$$sec^2\ A = tan^2\ A + 1,$$

we immediately obtain the corresponding geometric formula (without consulting TABLE I.), by introducing—not R instead of 1; for then we should have

$$sec^2\ A = tan^2\ A + R;$$

that is, a *square* equal to another *square* + a *right line,* which would be absurd,—but R^2, as in this case the resulting formula would be

$$sec^2\ A = tan^2\ A + R^2;$$

that is, a square equal to the sum of two squares, which is a rational equation.

Obs. 2. The *sines, cosines,* &c., heretofore spoken of, are called *natural sines, cosines,* &c. But these are now seldom registered in mathematical tables; instead of them we register their *logarithms,* which is found to be more convenient in practice. The logarithms of the *sines, cosines,* &c. of angles, are called the *logarithmic sines, cosines,* &c., of these angles; and being put opposite their corresponding angles in the tables, are used instead of the natural sines, &c, with great advantage, as will be seen hereafter.

We may calculate the logarithms of the natural *sines*, &c., from these sines themselves; but for this purpose it is found most eligible to take the numeral radius = 10,000,000,000, or 10^{10}, instead of 1, and to consider the natural *sines*, &c., as made up of submultiple parts of *this* numeral radius, in the same manner as we before considered them as made up of submultiple parts of 1 [*]. In fact 10^{10}, or *any* other number, may be looked upon as *the unit*, or whole, represented by 1, and therefore we may introduce 10^{10}, or the said number, as radius, into the formulæ of ARTS. 16 and 17, merely recollecting to *square* the new radius wherever 1 has been squared, though its square be not expressed, inasmuch as $1^2 = 1$.

N. B. When there is no especial purpose, such as the above, to be served, 1 is always chosen as numeral radius; because its *powers* being all = 1 disappear in computation, which is thereby much abbreviated.

LESSON V.

Trigonometry.

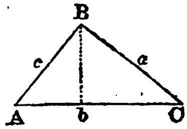
ART. 18. *In a triangle the sides are to each other as the sines of their opposite angles.*

Let ABC be any triangle, of which the three sides are represented severally by *a, b, c,* and the three corresponding opposite angles by A, B, C. Then

$$a : c :: \sin A : \sin C,$$

and $$a : b :: \sin A : \sin B,$$

and $$b : c :: \sin B : \sin C.$$



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[*] If 1 were taken as radius, the sines and cosines of almost all angles being less than the radius, would be fractions, and therefore would have their logarithms subtractive numbers (ART. 6°, LOG.) This, amongst others, is the reason we choose a number so much greater than 1 for radius as will render the sines and cosines of all but indefinitely small angles greater than unity, so that their logarithms will be additive numbers.

Dem. Represent the perpendicular from the vertex of any angle B on the opposite side by y. According to *Obs.* 2, **Lesson IV**, we shall have

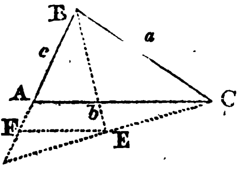
$$\frac{y}{c} = \sin A,$$

$$\frac{y}{a} = \sin C.$$

$$\therefore \sin A : \sin C :: \frac{y}{c} : \frac{y}{a} :: ay : cy :: a : c.$$

In like manner it is shown that $\sin A : \sin B :: a : b$, and $\sin B : \sin C :: b : c$, by drawing perpendiculars from the vertices of the other angles C and A to the other sides c and a. This, &c.

Art. 19. *In a triangle the tangent of half the sum of any two angles is to the tangent of half their difference, as the sum of the opposite sides is to their difference.*



Let ABC be a triangle, whose sides and angles are represented, as above, by a, b, c, and A, B, C. Then, $\tan\left(\frac{A+C}{2}\right) : \tan\left(\frac{A-C}{2}\right) :: a+b : a-b$.

Dem. Produce BA, which is not the greater of the two sides under consideration, until it be equal to the other BC; join CD, and let fall on it the perpendicular BE, which divides it equally at E, by **Art.** 139, **Geom.** The angle BCD = BDC (**Art.** 4, **Geom.**); therefore BDC = BCA + ACD. But by **Art.** 35, **Geom.**, BAC = BDC + ACD, $\therefore$ BAC = BCA + 2ACD, $\therefore$ ACD $= \frac{\text{BAC} - \text{BCA}}{2} = \frac{A - C}{2}$. Likewise, as BAC = BDC + ACD, adding the angle BCA to both, we get BAC + BCA = BDC + ACD + BCA = 2BDC = 2BCD $\therefore$ BCD $= \frac{\text{BAC} + \text{BCA}}{2} = \frac{A + C}{2}$.

Again: $AD = BD - BA = BC - BA$; $\therefore$ if AD be equally divided at F, $AF = \frac{AD}{2} = \frac{BC - BA}{2} = \frac{a-c}{2}$; and $2BF = 2BA + 2AF = BD + BA = BC + BA$, $\therefore BF = \frac{BC + BA}{2} = \frac{a+c}{2}$.

Finally: Joining FE, by Art. 161, Geom., FE is parallel to AC. Consequently $BE : bE :: BF : AF$, by Art. 110, Geom.; that is, representing the perpendiculars BE and bE respectively y and y', as also the line EC by x, we have

$$y : y' :: BF : AF,$$

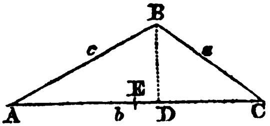
$$\therefore \frac{y}{x} : \frac{y'}{x} :: BF : AF :: \frac{a+c}{2} : \frac{a-c}{2} :: a+c : a-c.$$

But, by *Obs.* 2, Lesson IV, $\frac{y}{x} = tan\ BCD = tan\ \frac{A+C}{2}$, and $\frac{y'}{x} = tan\ ACD = tan\ \frac{A-C}{2}$; hence

$$tan\ \frac{A+C}{2} : tan\ \frac{A-C}{2} :: a+c : a-c.$$

In like manner it is shown that $tan\ \frac{A+B}{2} : tan\ \frac{A-B}{2} :: a+b : a-b$; and that $tan\ \frac{B+C}{2} : tan\ \frac{B-C}{2} :: b+c : b-c$. This, &c.

Art. 20. *In a triangle the cosine of any angle multiplied by twice the product of the sides which contain it, is equal to the sum of the squares of those sides diminished by the square of the third side.*



Let ABC be a triangle, whose sides and angles are represented as above by a, b, c, and A, B, C. Then, $cos\ A \times 2bc = b^2 + c^2 - a^2$.

Dem. Represent the perpendicular BD by y, and dividing AC equally at E, represent the intercept AD by x. The intercept DC will be represented by $(b-x)$. By ART. 47, GEOMETRY,

$$c^2 = y^2 + x^2, \text{ and } a^2 = y^2 + (b-x)^2,$$
$$\therefore\ c^2 - x^2 = a^2 - (b-x)^2,$$
$$\therefore\ c^2 - a^2 = x^2 - (b-x)^2 = -b^2 + 2bx,$$
$$\therefore\ 2bx = b^2 + c^2 - a^2,$$
$$\therefore\ 2bc \times \frac{x}{c} = b^2 + c^2 - a^2;$$

that is, as $\cos A = \dfrac{x}{c}$,

$$\cos A \times 2bc = b^2 + c^2 - a^2,$$

or, in another form,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

This, &c.

Obs. We are now in a condition to proceed to the measurement of *triangles*, or Trigonometry, properly so called. In a triangle there are three sides and three angles: certain of these being given in magnitude, determine the triangle both in form and magnitude. For,

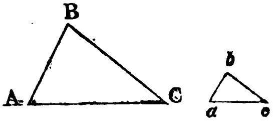
1°. Two sides and the included angle being given, the triangle can be of but one form and magnitude, by ART. 1, GEOM.

2°. Three sides being given, the triangle can be of but one form and magnitude, by ART. 7, GEOM.

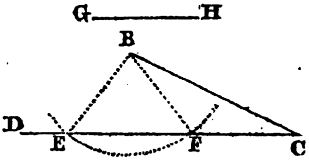
3°. Two angles and a side being given, the triangle can be of but one form and magnitude, by ART. 46, GEOM.

There are no other three of the above six parts which will absolutely and completely determine the triangle. For if the three *angles* be given, the triangle is determined indeed as to *form*, inasmuch as all triangles with three such angles must, by ART. 112, GEOM., have their sides respectively proportional to each other, and therefore be similar in figure. Thus, if ABC, *abc*, be two triangles having the three angles at A, B, C, equal respectively to those at *a*, *b*, *c*; then, by ART. 112, GEOM., AB : *ab* : : BC : *bc*, and AB : *ab*

: : AC : *ac*, and BC : *bc* : : AC : *ac*; so that, having their angles equal and their sides proportional, these triangles are similar in form. But they are, or at least may be, very unequal in *magnitude*, and therefore a triangle is not determinable from such data, completely as to form and magnitude, but partially as to form alone.



Again: if two sides and an angle *opposite* one be given, the triangle may be determinable neither as to form nor magnitude. For if BC, GH, be the two given sides, and BCD the one given angle, a circle may be described with B as a centre, and with a line = GH as a radius, which may meet CD in *two* points E and F, thus forming two different triangles EBC, FBC, each having two sides equal to BC, GH, and the given angle BCD; yet the triangles themselves will be, as in the figure, both unequal in magnitude and unlike in form.



It is evident, however, that in this case, although the triangle would not be determined absolutely, it would be determined *alternatively*; for it would be *either* EBC or FBC. These triangles agree so far that they have two sides BE, BC, equal respectively to two BF, BC, one angle BCE equal to one BCF, and two other angles BEC, BFC, *supplementary*, inasmuch as BEC = BFE, by ART. 4, GEOM., and BFE together with BFC is equal to two right angles (see DEF. VI.). We may therefore add another case, viz.:

4°. Two sides and an angle opposite one being given, the triangle can be of but one, or a second, form and magnitude.

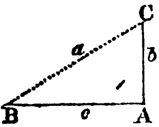
These are all the distinct data which can be made out by combining the six parts of a triangle in *threes*. It would not be determinable from a less number of parts, and to propose a greater would be superfluous. We shall therefore proceed to exhibit the method of computing triangles in these four cases.

LESSON VI.

Solution of Triangles.

For simplicity, and because such examples are most frequent, we shall first enter on the determination of *right-angled* triangles. In these we may consider one angle, viz. the right angle, which is $= 90^\circ$, as always given; therefore the whole triangle will be determinable from any other *two* parts being given.

Art. 21. *In a right-angled triangle given the two sides containing the right angle, the triangle itself may be determined.*



Let a, b, c, be respectively the three sides of the right-angled triangle; A, B, C, the corresponding opposite angles, A being the right one. Then, b, c, A, being given or known, a, B, C, may be determined.

Dem. For $a = \sqrt{b^2 + c^2}$; $\tan B = \frac{b}{c}$; and $\tan C = \frac{c}{b}$, as is evident from **Art.** 47, **Geom.**, and foregoing observations. This, &c.

Thus, suppose it were given that the two sides, b and c, about the right angle of a right-angled triangle were respectively equal to 3 and $3\sqrt{3}$ (feet, inches, or any other linear unit); then, we should have

$$b^2 = 3^2 = 9,$$
$$c^2 = (3\sqrt{3})^2 = 27,$$
$$\therefore\ a = \sqrt{9 + 27} = 6.$$

Likewise,

$$\tan B = \frac{1}{\sqrt{3}},\ \tan C = \sqrt{3}.$$

Here we have determined the third side, a, to be 6 (feet, inches, or whatsoever was the linear unit chosen). The two acute angles are also determined; because to find them from having their tangents, we have only to consult a trigonometrical table, and whatever angles in it have tangents of the values found, are equal to the angles computed. Thus, if 1 be the tabular radius, we shall see $\frac{1}{\sqrt{3}}$ registered as the tangent of 30°, and $\sqrt{3}$ as the tangent of 60°; for $\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{1}{\sqrt{3}}$, and $\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \sqrt{3}$. See TABLE IV.

Obs. 1. As the angles B and C are together equal to a right angle, having determined either, B, we may obtain the other immediately; for $C = 90^\circ - B$. Thus, if we find that B is $= 30^\circ$, C must be $= 90^\circ - 30^\circ = 60^\circ$.

There are also, in every trigonometrical question like the above, different methods of computing the sought quantities, some more convenient than others. But our object in this Treatise is merely to explain the nature and use of the science, without entering into details which would be unnecessary to the general reader.

In general, however, it may be said that these methods are used which seem best adapted for *logarithmic* computation.

Obs. 2. Instead of looking in a trigonometrical table for angles corresponding to the tangents so found, we more frequently take the logarithms of those values found for the tangents, and then consult a table such as we have described in *Obs.* 2, ART. 17. And this method of calculating by the help of logarithms is now chiefly pursued in works of science. Thus, if the sides of the above triangle were $b = 43$, and $c = 55$, we should have (using the tabular radius $r = 10^{10}$),

$$\tan B = r \cdot \frac{b}{c},$$

$$\therefore \log. \tan B = \log. r + \log. \frac{b}{c}, \text{ by ART. 3, LOG.};$$

$$= \log. r + \log. b - \log. c, \text{ by ART. 4, LOG.}$$
$$= \log. 10^{10} + \log. 43 - \log. 55.$$

But in a logarithmic table whose base = 10,

$$\begin{aligned}
\log.\ 10^{10} &= 10\\
\log.\ 43 &= 1{,}6334685\\
\therefore\ \log.\ 10^{10} + \log.\ 43 &= 11{,}6334685\\
\text{and, } \log.\ 55 &= 1{,}7403627\\
\therefore\ \log.\ 10^{10} + \log.\ 43 - \log.\ 55 &= 9{,}8931058,
\end{aligned}$$

which logarithm corresponds to an angle = $38^\circ\ 1'\ 8''$, $\therefore\ \mathrm{B} = 38^\circ\ 1'\ 8''$, and $\therefore\ \mathrm{C} = 51^\circ\ 58'\ 52''$.

Again,

$$a = \sqrt{b^2 + c^2};$$

therefore a may be found by extracting the square root of $43^2 + 55^2$. It is easier, however, to find the value of a by logarithms, to which the preceding formula is not adapted, being unresolvable into factors, and thereby preventing us from substituting the sum of their two logarithms for the logarithm of their product. We must, therefore, obtain the value of a by means of some other formula, viz.:

$$\cos \mathrm{B} = r\,\frac{c}{a}.$$

$$\therefore\ a = r\,\frac{c}{\cos \mathrm{B}},$$

$$\begin{aligned}
\therefore\ \log.\ a &= \log.\ r + \log.\ \frac{c}{\cos \mathrm{B}},\\
&= \log.\ r + \log.\ c - \log.\ \cos \mathrm{B},\\
&= 10 + 1{,}7403627 - \log.\ \cos 38^\circ 1' 8'';
\end{aligned}$$

but, in the table of logarithmic cosines, 9,8964202 corresponds to an angle of 38° 1′ 8″, and therefore we have

$$\begin{aligned}
\log\ a &= 10 + 1{,}7403627 - 9{,}8964202,\\
&= 1{,}8439425;
\end{aligned}$$

and, as 1,8439425 is the logarithm of the number 69, 81, &c., we find

$$a = 69,\ 81,\ \&c.$$

We give the preceding example of trigonometrical calculation by help of Logarithms, in order to evince the practical utility of the latter science, and illustrate the process by which we in general compute the sides and angles of a triangle from certain data. Investigations of the same kind are of no further use to the reader, and we shall therefore give merely the theoretical solutions for the future.

ART. 22. *In a right-angled triangle given the side opposite the right angle and one of the remaining sides, the triangle itself may be determined.*

Representing the sides and angles as above, then a, b, A, being given, c, B, C, may be determined.

DEM. For $c = \sqrt{a^2 - b^2}$, by ART. 47, GEOM.; and $\sin B = \frac{b}{a}$, $\cos C = \frac{b}{a}$,

In logarithms *, as

$$\sqrt{a^2 - b^2} = \sqrt{(a+b).(a-b)} = (a+b)^{\frac{1}{2}}.(a-b)^{\frac{1}{2}}, \therefore$$

$$\log. c = \log. (a+b)^{\frac{1}{2}}.(a-b)^{\frac{1}{2}} = \log. (a+b)^{\frac{1}{2}} + \log. (a-b)^{\frac{1}{2}};$$

$\therefore$ by ART. 6, LOG., $\log. c = \frac{1}{2} \log. (a+b) + \frac{1}{2} \log. (a-b)$.

And,

$$\log. \sin B = \log. b - \log. a,$$
$$\log. \cos C = \log. b - \log. a.$$

This, &c.

ART. 23. *In a right-angled triangle given the side opposite the right angle, and one acute angle, the triangle itself may be determined.*

Representing the sides and angles as above, then a, B, A, C, being given †, b, c, may be determined.

DEM. For, $\sin B = \frac{b}{a}$, $\therefore b = a . \sin B$; and $\frac{c}{a} = \cos B$, $\therefore c = a . \cos B$.

---

* To shorten computation we will use the ordinary radius 1, instead of 10^{10}, which is only necessary when we would solve the question arithmetically.

† One acute angle, as B, of a right-angled triangle, being given, both are given, inasmuch as $C = 90° - B$.

In logarithms,

$$\log. b = \log. a + \log. \sin \text{B},$$
$$\text{and, } \log. c = \log. a + \log. \cos \text{B}.$$

This, &c.

Art. 24. *In a right-angled triangle given either side about the right angle, and either acute angle, the triangle itself may be determined.*

Representing the sides and angles as above, then b, A, B, C, being given, a, c, may be determined.

Dem. For $\sin \text{B} = \frac{b}{a}$, $\therefore a = \frac{b}{\sin \text{B}}$; and $\frac{b}{c} = \tan \text{B}$, $\therefore c = \frac{b}{\tan \text{B}}$.

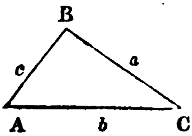
In logarithms,

$$\log. a = \log. b - \log. \sin \text{B},$$
$$\text{and, } \log. c = \log. b - \log. \tan \text{B}.$$

This, &c.

Triangles which are not right-angled may be solved, that is, determined from certain data, with little less facility than the above.

Art. 25. *In any triangle given two sides and the included angle, the triangle itself may be determined.*



Let a, b, c be respectively the three sides of the triangle, A, B, C the opposite corresponding angles. Then, any two sides, as a, b, and the included angle C, being known, the third side c, and the remaining angles A, B, may be determined.

Dem. By Art. 17,

$$\tan\left(\frac{\text{A}-\text{B}}{2}\right) = \tan\left(\frac{\text{A}+\text{B}}{2}\right) \cdot \frac{a-b}{a+b};$$

but as $\text{A} + \text{B} + \text{C} = 180°$, $\therefore \text{A} + \text{B} = 180° - \text{C}$, $\therefore \frac{\text{A}+\text{B}}{2} = 90° - \frac{\text{C}}{2}$. Consequently, subtracting half the given angle

C from a right angle, we obtain $\frac{A+B}{2}$, and therefore can find its tangent in the tables. Hence, as a and b are likewise given, we obtain the whole value of $\tan\frac{A-B}{2}$, from whence we get the corresponding angle $\frac{A-B}{2}$ by the tables.. But from half the sum $\frac{A+B}{2}$, and half the difference $\frac{A-B}{2}$, of the two angles A and B, we can determine both separately; as we have only to *add* half the difference to half the sum for the greater, and to subtract half the difference from half the sum for the lesser. Thus, if $C = 40^\circ$, then $\frac{A+B}{2} = 90^\circ - \frac{C}{2} = 70^\circ$; and if, from the above method of computation, we find, by the tables, that $\frac{A-B}{2} = 10$, then

$$\frac{A+B}{2} + \frac{A-B}{2} = A = 70^\circ + 10^\circ = 80^\circ,$$

$$\text{and, } \frac{A+B}{2} - \frac{A-B}{2} = B = 70^\circ - 10^\circ = 60^\circ\text{*}.$$

Again: having thus computed A and B, we may find c. For, by ART. 16,

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \therefore c = a \cdot \frac{\sin C}{\sin A};$$

---

* It is plain from the formula for $\tan\left(\frac{A-B}{2}\right)$, that, to compute it, we need not be given the absolute magnitude of the sides a, b, but merely their ratio. For suppose that ratio be $m : n$, i. e.

$$a : b :: m : n,$$
$$\therefore a+b : a-b :: m+n : m-n,$$
$$\therefore \frac{a-b}{a+b} = \frac{m-n}{m+n},$$

therefore we might use $\frac{m-n}{m+n}$ instead of $\frac{a-b}{a+b}$.

from which formula for c its value may be determined, as a, $\sin$ C, and $\sin$ A, are known quantities.

In logarithms,

$$\log. c = \log. a + \log. \sin \mathrm{C} - \log. \sin \mathrm{A}.$$

This, &c. *

Art. 26. *Given two angles of a triangle, and the side opposite either of them, the triangle itself may be determined.*

Representing the sides and angles as above, then any two angles, as A, B, and a side, as a, opposite either of these angles, A, being given, the third angle, C, and the remaining sides, b, c, may be found.

Dem. The third angle, C, is found by subtracting the sum of A and B from 180°; for $\mathrm{A} + \mathrm{B} + \mathrm{C} = 180°$, $\therefore \mathrm{C} = 180° - (\mathrm{A} + \mathrm{B})$. Likewise, by Art. 16,

$$\frac{b}{a} = \frac{\sin \mathrm{B}}{\sin \mathrm{A}}, \therefore b = a \,.\, \frac{\sin \mathrm{B}}{\sin \mathrm{A}};$$

$$\text{and, } \frac{c}{a} = \frac{\sin \mathrm{C}}{\sin \mathrm{A}}, \therefore c = a \,.\, \frac{\sin \mathrm{C}}{\sin \mathrm{A}}.$$

Hence, b and c are found in terms either given as a, or easily computable from trigonometrical tables, as $\sin$ A, $\sin$ B, $\sin$ C.

In logarithms,

$$\log. b = \log. a + \log. \sin \mathrm{B} - \log. \sin \mathrm{A}.$$

$$\text{and, } \log. c = \log. a + \log. \sin \mathrm{C} - \log. \sin \mathrm{A}.$$

This, &c.

Art. 27. *Given two angles of a triangle, and the side between them, the triangle itself may be determined.*

For two angles being given the third is determinable, as in Art. prec.; and the solution of this question will be thus reduced to the last.

Art. 28. *Given the three sides of a triangle, the triangle itself may be determined.*

Representing the sides and angles as above, then a, b, c being given, A, B, C may be found.

---

* c may be found immediately; because $\cos \mathrm{C} \times 2ab = a^2 + b^2 - c^2$, by Art. 20. In this formula, a, b, and $\cos$ C are known quantities, whence c may be found.

Dem. By Art. 18,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

From which value, in known terms, for *cos* A, we may find A itself from the tables. Hence, the triangle may be determined by Art. 23, as we know one of its angles and the including sides. This, &c.

Obs. We may determine B and C without the help of Art. 23, as we determined A, namely by the formulæ (Art. 20)

$$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

There are also various other methods of resolving this question, but one is sufficient to our purpose *.

Art. 29. *Given two sides of a triangle, and the angle opposite either of them, the triangle itself may be in some cases determined.*

Representing the sides and angles as above, then any two sides, as *a*, *b*, and the angle, as A, opposite either of them, *a*, being given, the triangle itself may be determined in two cases, viz. 1°, if the given side *not* opposite the given angle be less than the other given side; 2°, if the given side *not* opposite the given angle be less than the third side.

Dem. By Art. 18,

$$\frac{\sin B}{\sin A} = \frac{b}{a},\ \therefore \sin B = \frac{b}{a}\,.\sin A.$$

In logarithms,

$$\log.\sin B = \log.\sin A + \log.b - \log.a.$$

Consequently *sin* B is determined, the quantities *b*, *a*, and *sin* A being known. But as, by Art. 6, the sine is the same for two supplementary angles, we cannot determine to which of these the sine found belongs, unless, in some cases, from other considerations.

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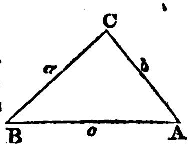
* It should be observed that the method here given is not adapted to logarithms, and therefore others of a more complex nature are used in preference.



1°, If b be less than a, then, by ART. 41, GEOM., B will also be less than A. Therefore, if A be acute, B will likewise be acute; and if A be not acute, B will be acute by ART. 38, GEOM. Hence, in this case B must be that supplementary angle, which is less than 90°.

Thus, if $b : a :: 1 : 2$, and $\sin A = \sqrt{3}$; then, $\sin B = \frac{\sqrt{3}}{2}$, which is the sine either of an angle equal 60°, or 120°; but as b is less than a the angle B must be acute, and therefore must be the lesser of these supplementary angles, i. e. 60°.

2°. If b be less than c, then by ART. 41, GEOM., B will be also less than C. Therefore, if C be acute, B will be likewise acute; and if C be not acute, B will be acute by ART. 38, GEOM. Hence, as before, though the value we find for *sin* B belongs indifferently to two supplemental angles, as we know from other considerations that B is acute, we know that it must be the lesser of these supplementary angles, and therefore can determine it.

Again: having computed B, we obtain C, because $C = 180° - (A + B)$; and having thus found C, we may obtain the third side c, of the triangle; for,

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \therefore c = a \cdot \frac{\sin C}{\sin A}.$$

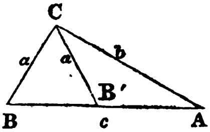
In logarithms,

$$\log. c = \log. a + \log. \sin C - \log. \sin A.$$

This, &c.

Obs. Even without these considerations abovementioned, the triangle can be determined to *one of two* triangles; for *sin* B can only belong to one of two angles, viz. supplementary ones, as was explained in page 78. Thus, if $b : a :: 2 : 1$, and $\sin A = \frac{1}{4}$; then, $\sin B = \frac{1}{2}$, which is the sine of an angle equal 30° or 150°, $\therefore$ B must $= 30°$ or 150°. Hence, according as we take either of these values for B, the third angle C will have a corresponding value, and ∴ also the third side, c, which depends on the value we assign to C in the formula above,

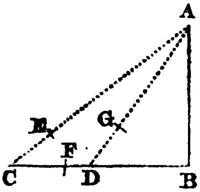
$$c = a \cdot \frac{\sin C}{\sin A}.$$



Consequently, with the given sides b, a, and the given angle A, as each other part of the required triangle may have two values, and only two, the triangle itself may be one of two, CBA or CB′A, as in the annexed figure, and must be one of these, though which it is impossible to determine.

APPLICATION OF TRIGONOMETRY TO PRACTICE.

I. *The perpendicular height of an object may be determined though the foot be inapproachable.*



Let A be any object, AB its perpendicular height from the surface on which we stand. Then, the height AB may be computed without approaching the perpendicular AB.

DEM. The surface CDB on which we stand being supposed plane, or perfectly flat, draw the right line CD, of any convenient length, and measure it. At the point C direct a straight ruler CE until it covers the object A, and measure the angle ECF between CE and CD. In the same manner, at the point D, measure the angles CDA, ADB, by directing the ruler DG to cover the object A *. These things being done, we have, in the triangle ADC, the angles ACD and ADC by measurement; therefore we know the angle CAD, because CAD = 180° — (ACD + ADC). Consequently, as by ART. 18,

---

* A *quadrant*, or other instrument, is generally used to measure these angles with exactness; but we must then allow for the height of the instrument from the ground.

$$\text{AD} : \text{CD} :: \sin \text{ACD} : \sin \text{CAD},$$

$$\therefore \text{AD} = \text{CD} \times \frac{\sin \text{ACD}}{\sin \text{CAD}};$$

and therefore the line AD is determined in known quantities CD, *sin* ACD, and *sin* CAD. Now, as AB is perpendicular to CDB, we have, in the triangle ADB,

$$\frac{\text{AB}}{\text{AD}} = \sin \text{ADB},$$

$$\therefore \text{AB} = \text{AD} \times \sin \text{ADB};$$

hence AB is determined in the known quantities AD, *sin* ADB, which was the thing required.

Obs. By the same means we can determine the perpendicular height of any object, such as a cloud, &c., it being of no importance in the preceding investigation whether the object A is supported on the plane, or suspended in the air. Likewise, the perpendicular height of a tower, or other elevation, may be computed, though the foot be not only inapproachable but *invisible;* for it is sufficient if the top alone be observable. We have only to measure the line CD, and the angles ACD, CDA, ADB, and proceed as above.

II. *The distance of an elevated object may be determined though its foot be inapproachable.*

Let A (in the above figure) be any object. Then, its distance, CA, may be determined.

DEM. Measuring the line CD, and the angles ACD, CDA, we know the angle CAD as before; and

$$\text{AC} : \text{CD} :: \sin \text{CDA} : \sin \text{CAD},$$

$$\therefore \text{AC} = \text{CD} \times \frac{\sin \text{CDA}}{\sin \text{CAD}};$$

hence AC is determined in known quantities CD, *sin* CDA, and *sin* CAD. Which was, &c.

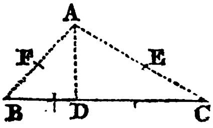
Obs. 1. The distance, CB, along the plane, is also determinable. For,

$$\frac{\text{CB}}{\text{AC}} = \cos \text{ACD},$$

$$\therefore \text{CB} = \text{AC} \times \cos \text{ACD}.$$

Obs. 2. The above distances AC, CB, are determinable, as is evident, though the extremity A of the perpendicular AB be alone visible. And in this manner we may compute the distance of a cloud, pinnacle, battlement, or any other elevated object, from whatever point of view we choose.

III. *The distance of an object on the ground may be determined, though it be inapproachable.*



Let A be any object on the plane surface BAC, which we suppose to represent the ground. Then, the distance AB, from the object, A, to the place of the spectator, B, may be computed, without approaching the object.

DEM. On the surface BAC draw the right line BC of any convenient length, and measure it. At the points B and C direct the rulers BF and CE, and measure the angles ABC, ACB, between them and the line BC. Hence, we know the angle BAC, for BAC = 180° — (ABC + ACB). Consequently, as

$$\text{AB} : \text{BC} :: \sin \text{ACB} : \sin \text{BAC},$$

$$\therefore \text{AB} = \text{BC} \times \frac{\sin \text{ACB}}{\sin \text{BAC}};$$

hence, the distance AB is determined in known quantities BC, *sin* ACB, and *sin* BAC. Which, &c.

Obs. By this means we may compute the breadth of a river, or chasm, from one side of it. For we have only to draw BC along the brink, and determine the distance AB from one extremity, B, of this line to some object, A, on the opposite brink, as in the above example. And if we wish to compute the direct breadth, AD, we may find it thus:

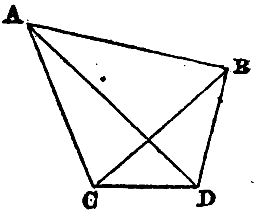
$$\frac{\text{AD}}{\text{AB}} = \sin \text{ABD},$$

$$\therefore \text{AD} = \text{AB} \times \sin \text{ABD};$$

hence, AD is determined in quantities, AB and *sin* ABD, already computed or measured.

IV. *The distance between two inapproachable objects may be determined.*

Let A, B, be two inapproachable objects. Then, their distance AB may be determined.



Dem. Draw upon the ground the right line CD of any convenient length, and measure it. By the help of rulers along CA, CB, DA, and DB, let the angles ACD, BCD, ADC, CDB, be measured. Now, in the triangle ACD, the angle CAD $= 180^\circ - (\mathrm{ACD} + \mathrm{ADC})$, and is therefore known. Also, in the triangle BCD, the angle CBD $= 180^\circ - (\mathrm{BCD} + \mathrm{CDB})$, and is therefore known. Consequently, as

$$\mathrm{CA} : \mathrm{CD} :: \sin \mathrm{ADC} : \sin \mathrm{CAB},$$

$$\text{and, } \mathrm{CB} : \mathrm{CD} :: \sin \mathrm{CDB} : \sin \mathrm{CBD},$$

$$\therefore\ \mathrm{CA} = \mathrm{CD} \times \frac{\sin \mathrm{ADC}}{\sin \mathrm{CAB}},$$

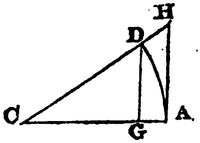
$$\text{and, } \mathrm{CB} = \mathrm{CD} \times \frac{\sin \mathrm{CDB}}{\sin \mathrm{CBD}};$$

and therefore CA, CB, are determined in known quantities. But the angle ACB $=$ ACD $-$ BCD, and is therefore known. Hence, in the triangle ACB we have the angle ACB, and the sides about it CA, CB, to determine the base AB; which may be done by Art. 25. Which, &c.

ADDENDUM TO TRIGONOMETRY.

One or two principles which the Student will find extremely serviceable, but which would have interfered with the regularity of our system, may find place here in the form of addenda.

Art. 30. *In a very small arch the sine is nearly equal to the tangent.*



Let AD be any arch, DG its sine, AH its tangent. Then, if AD be very small, DG is nearly equal to AH.

Dem. DG : AH : : CG : CA, by Art. 112, Geom.; but when AD is very small, its cosine, CG, is nearly = to the radius CA, and ∴ also the sine DG is, in this case, nearly equal to the tangent AH. This, &c.

Obs. Hence in very small arches the sine and tangent may be considered as equal; as also the sine, tangent, and *arch itself*, which latter is between the two former, and may, in these cases, be taken as a right line.

Art. 31. *The circumference of a circle is nearly equal to three times and a seventh its diameter.*

Dem. By Table II. we have calculated the sine of an arch = 15°, and find it to be

$$\sin 15^\circ = \text{R}\sqrt{\tfrac{1}{2} - \frac{\sqrt{3}}{4}}.$$

And by formula 9, Table I, we have

$$2 \sin 7\tfrac{1}{2}^\circ \times \cos 7\tfrac{1}{2}^\circ = \text{R} \times \sin 15^\circ,$$

$$= \text{R}^2 \sqrt{\tfrac{1}{2} - \frac{\sqrt{3}}{4}},$$

$$\therefore 2 \sin 7\tfrac{1}{2}^\circ \times (\text{R}^2 - \sin^2 7\tfrac{1}{2}^\circ) = \text{R}^2 \sqrt{\tfrac{1}{2} - \frac{\sqrt{3}}{4}}.$$

Consequently, from this equation we can get the value of *sin* $7\frac{1}{2}^\circ$, or of the sine of *half* the first angle. And, in the same manner, by applying the same formula, we can get the value of the sine of *half* the angle $= 7\frac{1}{2}^\circ$, i. e., we can get the value of the sine of $3^\circ\ 45'$. By thus proceeding continually, we can at length get the value of the sine of an angle as small as we choose. Now the value of the sine of an angle $= 1'$ has been actually computed, and it has been found that, in decimals,

$$\sin 1' = \text{R} \times \cdot00029, \&c.$$

But by *Obs.* ART. prec., *sin* $1'$ = arch of $1'$; therefore multiplying the above value for an arch $= 1'$ by 60, we shall get the value, or length of an arch $= 60' = 1^\circ$, viz.:

$$\begin{array}{r} \text{R} \times \cdot00029 \\ 60 \\ \hline \text{R} \times \cdot01740 \end{array}$$

And multiplying this by 360, we shall get the length of an arch $= 360^\circ$ = the circumference, viz.

$$\begin{array}{r} \text{R} \times \cdot01740 \\ 360 \\ \hline 104400 \\ 5220 \\ \hline \text{R} \times 6\cdot26400 \end{array}$$

Hence, we find that the circumference is = R × 6·26400, i. e. about *six* times and *two-sevenths* the radius, or *three* times and *one-seventh* the diameter. This, &c.

Obs. The ratio of the circumference of a circle to its diameter, investigated by methods still more accurate than that we have exhibited (which is only given here for the sake of illustration), is found to be nearly equal to the ratio of 3·141592 : 1.

Mathematicians have chosen the Greek letter π (p) to stand for the above number; and in all scientifical treatises where this symbol π occurs, it is generally understood to represent the ratio of the circumference of a circle to its diameter; hence it will be useful to recollect the formula

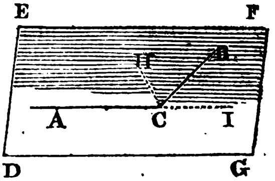
$$\pi = \frac{circumference}{diameter} = 3{\cdot}141592.$$

SPHERICAL TRIGONOMETRY.

Prefatory Principles.

A FEW principles of Solid Geometry essential to the following Treatise, and useful in all Science, may be well introduced here: so very few, however, are necessary for our Series that it would be absurd to collect them into a totally separate work.

ARTICLE 1. *One portion of a right line cannot be in a plane, and another portion out of it.*



Let ACB be a right line, one part of which, AC, is in the plane DEFG. Then, no other part of the line can be out of the plane DEFG.

DEMONSTRATION. *Suppose* the part CB to lie out of the plane DEFG. Produce the right line AC, in the plane DEFG, to any point I, and let a plane pass through the points A and I. Then the line AI is in this plane *. Also let this plane be turned about AI until it pass through the point B; and as the points C, B, are in this new plane, the right line CB is also in this plane. Now draw CH in this plane, making, with CI, a greater angle than BCI: consequently, by ART. 9, GEOM., the angles ACH and HCI are together = to two right angles. But as the line ACB is likewise *supposed* a right one, the angles

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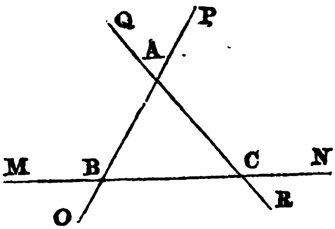
* AXIOM. A plane being a perfectly even surface, the right line joining any two points in it must lie wholly in that plane (else the right line itself could not be perfectly even between these points); and may also be produced to any length in that plane.

ACH and HCB together would be also equal to two right angles. Therefore the angle HCI would be equal to the angle HCB, the greater to the less,—which is impossible. Hence the above supposition is false; CB does not lie out of the plane DEFG. This, &c.

ART. 2. *Three right lines which meet one another are in the same plane.*



Let the three right lines MN, OP, QR, meet one another in the points A, B, C. Then, these three right lines are in one and the same plane.

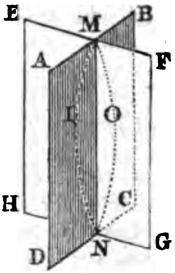
DEM. Conceive any plane passing through the line BC to be so turned about BC, so as to pass through the point A; the right lines BA, CA, having respectively the two points B and A, C and A, in this plane, must themselves lie in it. Hence, the three right lines BC, BA, CA, being in the same plane, the three right lines MN, OP, QR, would likewise be in it, by ART. preceding. This, &c.

Observation 1. Every triangle is in one and the same plane.

Obs. 2. *Two* right lines which meet one another, are in the same plane. For, if OP, QR, be the lines (figure preceding), conceive a plane passing through any point B in the one to be so turned as to pass through any point C in the other, and these points B and C to be joined by a right line in that plane. Hence, by ART. preceding, the three right lines BC, BA, CA, are in the same plane, and therefore also the two right lines BA and CA, or the whole lines OP and QR.

This truth might have been also proved in the same way as the ARTICLE from which we have deduced it.

ART. 3. *If two planes cut one another their line of intersection will be a right one.*

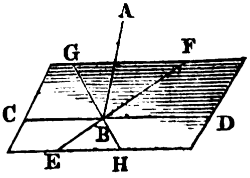


Let ABCD, EFGH, be two planes cutting one another in the line MN. Then, this line MN is a right one.

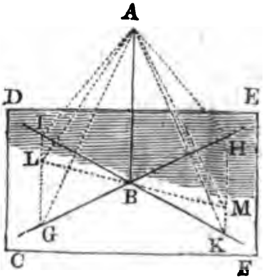
DEM. *Suppose* the line MN, which is common to the two planes, not to be a right line. In this case the points M and N might be joined by a right line MIN in the plane ABCD; and, as this right line is supposed different from MN, it is not in the plane EFGH, and there might be *another* right line, MON, drawn in the plane EFGH, between the same points M and N. Consequently, on this supposition, the two right lines MIN, MON, would enclose a space, which is impossible. Hence, the supposition is false; that is, the line of intersection MN is a right one. This, &c.

In common speech we say that a line is perpendicular to a plane when it *leans* no more to one side than another; this notion, stated a little more accurately, becomes the geometrical definition of a perpendicular to a plane.

DEF. I. A right line is said to be *perpendicular to a plane* when it is perpendicular to all the right lines in that plane which pass through the point where the line meets the plane.



Thus, if AB be a right line drawn to the plane CGFDHE, so that all the right lines in that plane which pass through the point B, such as CD, EF, GH, are perpendicular to AB, then AB is said to be perpendicular to the plane CGFDHE.



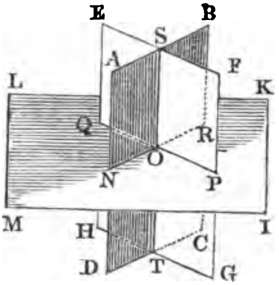
ART. 4. *If at the point where two right lines intersect, a right line stand perpendicular to both, it shall also be perpendicular to the plane in which they lie.*

Let AB be a right line perpendicular to both the right lines GH, IK, at their point of intersection, B. Then, AB is perpendicular to the plane CDEF, in which these lines GH, IK, lie.

DEM. Through the point B draw *any* right line LM, in the plane CDEF. Take the portions BG, BI, BH, BK, equal to each other; and draw the right lines AG, AI, AH, AK. In the four triangles thus formed, ABG, ABI, ABH, ABK, the four angles ABG, ABI, ABH, ABK, are granted equal, being all right angles. Likewise the side AB is common, and therefore as the sides BG, BI, BH, BK, have been taken equal, the four bases AG, AI, AH, AK, are equal, by ART. 1, GEOMETRY.

By ARTS. 1 and 2, GEOM., as the sides BI, BG, are equal respectively to the sides BH, BK, and the contained angles IBG, HBK, being vertically opposite, are also equal, therefore the angles BIL, BKM, are equal; and also the bases IG, HK. Consequently, the three sides AG, AI, IG, of the triangle AIG having been proved respectively equal to the three sides AH, AK, HK, of the triangle AHK, the angle AIL opposite AG is equal to the corresponding angle AKM opposite AH. But in the triangles IBL, KBM, the vertically opposite angles IBL, KBM, are equal; as also the angles BIL, BKM; and the sides BI, BK: therefore, by ART. 46, GEOM., IL = KM, and BL = BM. Thus, in the triangles AIL, AKM, we have shown that AI = AK, IL = KM, and the angle AIL = AKM; wherefore, by ART. 1, GEOM., AL = AM. Hence, as in the triangles ABL, ABM, it has been proved that AL = AM, and BL = BM, the side AB being common, we have the angle ABL = ABM, by ART. 6, GEOM.; that is, AB is perpendicular to LM. In the same manner it may be demonstrated that AB is perpendicular to every other right line in the plane CDEF; whence, by DEF. preceding, AB is perpendicular to the plane itself. This, &c.

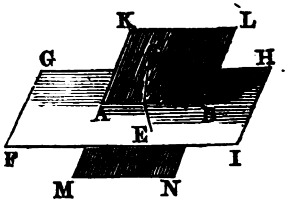
Again: in common practice the mutual inclination of two planes, which meet or cut each other, would be considered as rightly measured by cutting them with a third plane on which both would stand *upright*, and then taking the angle between the lines where the latter plane cuts the two former.



Thus, if ABCD, EFGH, were two planes cutting each other in the line ST, and IKLM, a third plane cutting the former in the lines NR, QP, so that they stood upright upon it, then we should say that the inclination of the planes ABCD, EFGH, was rightly measured by the inclination of the lines NO, OP, to each other.

If we now consider that the line of intersection, SO, must also be *upright* on the plane IKLM, that is, perpendicular to the lines NO, OP, or they perpendicular to it, the following definition will appear to spring very naturally from the above popular method of estimating the inclination of two planes to each other.

DEF. II. When two planes cut each other, the *angle between them,* or their mutual inclination, is measured by the angle between two right lines drawn in those planes from the same point of their line of intersection, and perpendicular to it.



Thus, if MKLN, FGHI, be two planes which intersect in the right line AB, and if DC, DE, be right lines drawn in the planes MKLN, FGHI, respectively, from the same point, D, and both perpendicular to AB; then, the angle which the said planes make with each other is measured by the angle between these right lines DC and DE. Any *one* such angle may be taken to measure the angle between the planes, inasmuch as their mutual inclination is evidently the same throughout their whole common length, AB. [NOTE C.]

DEF. III. Two planes are *perpendicular to each other* when the angle between them is measured by a right angle. Thus, in the above figure, if the angle between the perpendiculars DC, DE, be a *right* angle, the planes MKLN, FGHI, are said to be perpendicular to each other.

LESSON I.

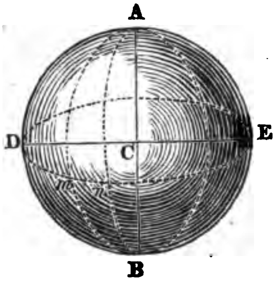
Of the Sphere.

Definition IV. Any figure made where a solid is cut by a plane surface, is called a *Section* of that solid.

Thus, if a die be cut by a plane parallel to any side of the die, each of the figures made where the plane passes is a section of the die, namely a square.

Obs. A diameter of a circle remaining fixed, if the circle be made to revolve about this diameter, it is evident that the circumference will describe a surface which is every where equally distant from the centre of the circle; and that if the circle revolve progressively, the surface described will at length return into itself, so as totally to include a solid.

Thus, if AB be any diameter of a circle ADBE, and if AB remain fixed, there will be a surface described by the progressive revolution of ADBE round AB, through the points *m*, *n*, E. And every point of the circle being equally distant from the centre C, therefore also every point of the surface described by the revolution of that circle will be equally distant from the same centre.



Def. V. A *sphere* is a solid figure bounded by one surface, such, that all right lines drawn from it to one and the same point within the figure are equal to one another.

Def. VI. In a sphere, the point from which the surface is every where equally distant, is called the *centre* of the sphere[*].

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[*] Compare the above *Observation* and *Definition* with *Observation* and *Def.* 2. Problems, of our Geometry. The *Sphere* is to Solids

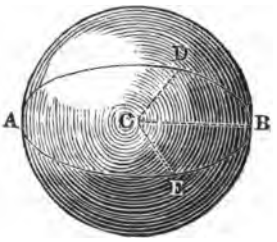
Obs. 1. It is plain that the centre of the sphere is also the centre of the revolving circle.

Def. VII. A right line drawn from the centre of a sphere, and terminated in the surface, is called a *Radius* of the sphere.

Def. VIII. A right line drawn through the centre of a sphere, and terminated both ways in the surface, is called a *Diameter* of the sphere.

Art. 5. *Every section of a sphere, made by a plane, is a circle.*

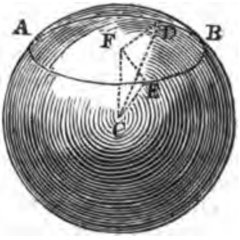
1



Let ADBEA be a plane section of a sphere. Then, ADBEA is a circle.

Dem. *First;* If the plane pass through the centre, C, of the sphere, as in figure 1. Draw right lines, CD, CE, CB, &c., from the centre to all points, D, E, B, &c., of the boundary of the section; and as all these right lines are drawn from the centre to the surface of the sphere, they are all equal, by Def. VII. Hence, by Def. 2, Geom., the bounding line of the section ADBEA is a circle.

2



Secondly; If the plane do not pass through the centre, C, of the sphere, as in figure 2. Draw the right line CF perpendicular to the plane of the section ADBEA; and also a right line, CD, CE, to any points, D, E, &c., of the boundary of the section. Join FD and FE. By Def. II. CF is perpendicular to FD and FE; therefore, by

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what the Circle is to Plane figures. Also, it is sufficiently evident that the centre of a sphere must be *within* the figure, and that a sphere has but *one* centre, which principles we therefore assume; or they may be proved in the same manner as Arts. 48 and 49, Geometry.

ART. 47, GEOM., $CD^2 = CF^2 + FD^2$, and also $CE^2 = CF^2 + FE^2$. Consequently, as $CD = CE$ by DEF. VI, $CF^2 + FD^2 = CF^2 + FE^2$, $\therefore FD^2 = FE^2$, $\therefore FD = FE$. Hence, all the right lines, such as FD, FE, &c., drawn from the point F to the boundary of the section, being equal, this boundary is a circle, by DEF. 2, GEOM. This, &c.

DEF. IX. A *great* circle of a sphere is that made by a plane passing through the centre of the sphere.

ART. 6. *All great circles of the same sphere are equal to each other.*

Because their respective radii, being all radii of the same sphere, are equal.

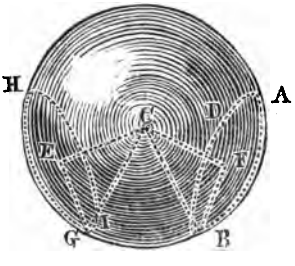
ART. 7. *Two great circles of the same sphere divide each other into two equal parts.*

Because the intersection of their planes is a diameter of the sphere, and a common diameter of both circles; which therefore divides both into equal parts.

Obs. Hence *all* great circles of the same sphere divide each other into two equal parts.

DEF. X. A *lesser* circle of a sphere is that made by a plane which does not pass through the centre of the sphere.

ART. 8. *All lesser circles equally distant from the centre of the sphere are equal to each other.*



Let ABD, GHI, be two small circles equally distant from the centre C (that is, having the perpendiculars CF, CE, drawn to them from the centre, equal). Then, the circle ABD, GHI, are equal to each other.

DEM. Draw CB, CG, from the centre C, to any points B, G, in the circumferences ABD, GHI; and join FB, EG. Consequently, as CF, CE, are, by DEF. I, perpendicular to FB, EG, we have, by ART. 47, GEOM., $CB^2 = CF^2 + FB^2$, and also $CG^2 = CE^2 + EG^2$. But $CB = CG$, being radii of the sphere, $\therefore CF^2 + FB^2 = CE^2 + EG^2$, or as $CF = CE \therefore FB = EG$. Hence, as the radii of the two circles ABD, GHI, are equal, the circles themselves must be equal. (Art. 81, Geom.) This, &c.

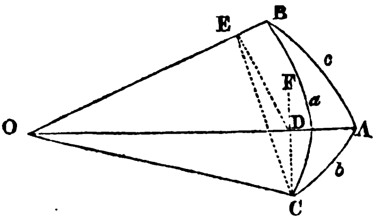
Def. XI. A *spherical triangle* is a spherical surface bounded by three arcs of great circles.

Obs. 1. The three arcs which bound a spherical triangle are called its *sides;* and are usually considered to be each less than a semi-circumference.

Obs. 2. The three angles contained between the planes of the three great circles whose arcs form a spherical triangle, are called the *angles of the triangle.*

Obs. 3. A spherical triangle is said to be *right-angled* when any of its angles is a right angle. But, unlike a plane right-angled triangle, it may have two or three of its angles *each* a right angle.

Art. 9. *In any right-angled spherical triangle, the product of the sine of the side opposite the right angle and the sine of either remaining angle, is equal to the sine of the side opposite the latter angle.*



Let ABC be a triangle, formed on the surface of a sphere whose centre is O, by the planes ABO, ACO, CBO, of three great circles, whose arcs AB, AC, CB, form the three sides of the triangle. Also, let the planes ABO, ACO, be perpendicular, that is, let the angle BAC of the triangle be a right one. Then, calling the three angles, BAC, ABC, ACB, of the triangle (that is, the three angles under the planes ABO and ACO, ABO and CBO, ACO and CBO), A, B, C, respectively, and the sides BC, AC, AB, opposite these angles respectively, a, b, c; we shall have

$$\sin a \times \sin B = \sin b, \text{ and } \sin a \times \sin C = \sin c.$$

Dem. From the vertex of the angle C draw the right line CD perpendicular to OA; and from the point D draw the right line DE perpendicular to OB. Join EC by a right line. Now, CD being perpendicular to OA, is perpendicular to the plane ABO; because drawing in this plane the right line DF at right angles to OA, CD will then be perpendicular to DO by construction, and to DF, by DEF. III, inasmuch as the planes ABO, ACO, are perpendicular to each other: consequently, by ART. 1, CD is perpendicular to the plane ABO. But CD being perpendicular to the plane ABO must be also perpendicular to the line DE, by DEF. I; and therefore CDE is a right-angled triangle. Hence, in the right-angled triangles CDO, DEO, CDE, we have, by ART. 47, GEOMETRY,

$$\mathrm{OC}^2 = \mathrm{OD}^2 + \mathrm{CD}^2;$$
$$\text{and } \mathrm{OD}^2 = \mathrm{OE}^2 + \mathrm{DE}^2;$$
$$\text{and } \mathrm{EC}^2 = \mathrm{ED}^2 + \mathrm{CD}^2, \therefore \mathrm{EC}^2 - \mathrm{ED}^2 = \mathrm{CD}^2.$$

Putting the values for OD^2 and CD^2 found from the last two equations into the first, we get

$$\mathrm{OC}^2 = \mathrm{OE}^2 + \mathrm{EC}^2;$$

consequently, by ART. 132, GEOM., the triangle OEC is also right-angled at E. Thus, DE and CE being drawn respectively in the planes ABO, CBO, perpendicular to OB, the intersection, we have the angle DEC equal to the angle between the planes, by DEF. II, that is, to the angle B. Likewise, in the right-angled triangle CDE, $\sin \mathrm{DEC} = \sin \mathrm{B} = \frac{\mathrm{CD}}{\mathrm{CE}}$. But, $\frac{\mathrm{CD}}{\mathrm{CO}} = \sin \mathrm{DOC} = \sin$ (of corresponding arch) $\mathrm{AC} = \sin b, \therefore \mathrm{CD} = \mathrm{CO} \times \sin b$; also $\frac{\mathrm{CE}}{\mathrm{CO}} = \sin \mathrm{COE} = \sin$ (of corresponding arch) $\mathrm{BC} = \sin a, \therefore \mathrm{CE} = \mathrm{CO} \times \sin a$. Hence, $\sin \mathrm{B} = \frac{\mathrm{CD}}{\mathrm{CE}} = \frac{\mathrm{CO} \times \sin b}{\mathrm{CO} \times \sin a}, \therefore \sin a \times \sin \mathrm{B} = \sin b$.

Instead of beginning with the vertex of angle C, we might as well have begun from that of angle B, and the process would have been exactly similar, only that for angle B we should have had angle C, and for b the side opposite B, we should have had c the side opposite C. Making these substitutions in the above formula, we obtain, without going through another demonstration, $\sin a \times \sin \mathrm{C} = \sin c$.

These, &c.

Art. 10. *In any right-angled spherical triangle, the tangent of either side adjacent to the right angle is equal to the product of the tangent of the side opposite the right angle and the cosine of the angle between these sides.*

In the above figure, representing the sides and angles as before, we shall have

$$\tan c = \tan a \times \cos \mathrm{B}, \text{ and } \tan b = \tan a \times \cos \mathrm{C}.$$

Dem. In the right-angled triangle CDE, cos DEC, or cos B, $= \frac{\mathrm{DE}}{\mathrm{CE}}$. But, $\frac{\mathrm{DE}}{\mathrm{OE}} = \tan \mathrm{DOE} = \tan$ (of arch) $\mathrm{BA} = \tan c$, $\therefore$ $\mathrm{DE} = \mathrm{OE} \times \tan c$; also, $\frac{\mathrm{CE}}{\mathrm{OE}} = \tan \mathrm{COE} = \tan$ (of arch) $\mathrm{BC} = \tan a$, $\therefore \mathrm{CE} = \mathrm{OE} \times \tan a$. Hence, $\cos \mathrm{B} = \frac{\mathrm{OE} \times \tan c}{\mathrm{OE} \times \tan a}$, $\therefore$ $\tan c = \tan a \times \cos \mathrm{B}$.

As explained in last Article, had we begun the construction in the figure from B instead of C, we should have obtained an exactly similar formula to the above, the angle C taking the place of B, and the side b that of c; that is to say, without further demonstration, $\tan b = \tan a \times \cos \mathrm{C}$. This, &c.

Art. 11. *In any right-angled spherical triangle, the cosine of the side opposite the right angle is equal to the product of the cosines of the remaining sides.*

In the last figure, retaining the same notation, we shall have

$$\cos a = \cos b \times \cos c.$$

Dem. $\operatorname{Cos} a = \cos$ (of arch) $\mathrm{BC} = \cos \mathrm{BOC} = \frac{\mathrm{OE}}{\mathrm{OC}}$. But $\frac{\mathrm{OE}}{\mathrm{OD}} = \cos \mathrm{DOE} = \cos$ (of arch) $\mathrm{BA} = \cos c$, $\therefore \mathrm{OE} = \mathrm{OD} \times \cos c$; also, $\frac{\mathrm{OD}}{\mathrm{OC}} = \cos \mathrm{DOC} = \cos$ (of arch) $\mathrm{AC} = \cos b$, $\therefore \mathrm{OC} = \frac{\mathrm{OD}}{\cos b}$. Hence, $\cos a = \frac{\mathrm{OD} \times \cos c}{\frac{\mathrm{OD}}{\cos b}} = \cos b \times \cos c$. This, &c.

Obs. There are some other properties of a right-angled spherical triangle of great utility in the sciences. But instead of demonstrating these geometrically in separate Articles, it will be shorter, easier, and therefore perhaps clearer, to deduce them algebraically from the principles just established, which may be done in one Article, as follows:

ART. 12. First:

$$\text{By ART. 9, } \sin B = \frac{\sin b}{\sin a},$$

$$\text{and by ART. 10, } \cos C = \frac{\tan b}{\tan a},$$

$$\text{hence, by division, } \frac{\sin B}{\cos C} = \frac{\sin b \times \tan a}{\sin a \times \tan b}.$$

But, as $\tan a = \frac{\sin a}{\cos a}$, and $\tan b = \frac{\sin b}{\cos b}$, the last formula becomes

$$\frac{\sin B}{\cos C} = \frac{\sin b \times \sin a \times \cos b}{\sin a \times \cos a \times \sin b}$$

$$= \frac{\cos b}{\cos a}, \text{ or, by ART. 11,}$$

$$= \frac{1}{\cos c}.$$

Hence we obtain the new formula

$$\sin B \times \cos c = \cos C.$$

In like manner, by dividing the other formulæ in ARTS. 9 and 10, *viz.*

$$\sin C = \frac{\sin c}{\sin a}, \text{ and } \cos B = \frac{\tan c}{\tan a},$$

we obtain the new formula

$$\sin C \times \cos b = \cos B,$$

which is perfectly symmetrical with the other, and might have been deduced from it at once, by changing the letters B, c, C, into C, b, B, respectively.

Again: by dividing these new formulæ, we get

$$\frac{\sin \mathrm{B} \times \cos c}{\cos \mathrm{B}} = \frac{\cos \mathrm{C}}{\sin \mathrm{C} \times \cos b};$$

but, as $\frac{\sin \mathrm{B}}{\cos \mathrm{B}} = \tan \mathrm{B}$, and $\frac{\cos \mathrm{C}}{\sin \mathrm{C}} = \operatorname{cotan} \mathrm{C}$,

this equation becomes

$$\tan \mathrm{B} \times \cos c = \frac{\operatorname{cotan} \mathrm{C}}{\cos b},$$

$$\therefore \tan \mathrm{B} \times \cos c \times \cos b = \operatorname{cotan} \mathrm{C};$$

or by ART. 11,

$$\tan \mathrm{B} \times \cos a = \operatorname{cotan} \mathrm{C},$$

which is a third new formula.

Again: inasmuch as cotan C $= \frac{\cos \mathrm{C}}{\sin \mathrm{C}}$, we get from this third formula,

$$\tan \mathrm{B} \times \cos a = \frac{\cos \mathrm{C}}{\sin \mathrm{C}}.$$

But, by ART. 10, $\cos \mathrm{C} = \frac{\tan b}{\tan a}$; and by ART. 9, $\sin \mathrm{C} = \frac{\sin c}{\sin a}$; consequently the above equation becomes

$$\tan \mathrm{B} \times \cos a = \frac{\tan b \times \sin a}{\tan a \times \sin c}, \text{ or as } \frac{\sin a}{\tan a} = \cos a,$$

$$= \frac{\tan b \times \cos a}{\sin c};$$

$$\therefore \tan \mathrm{B} \times \sin c = \tan b.$$

In like manner, by the third formula we have

$$\operatorname{cotan} \mathrm{C} = \tan \mathrm{B} \times \cos a, \text{ or, as above,}$$

$$= \frac{\tan b \times \cos a}{\sin c}, \text{ i. e. by ART. 11,}$$

$$= \frac{\tan b \times \cos b \times \cos c}{\sin c};$$

but $\tan b \times \cos b = \sin b$, and $\dfrac{\cos c}{\sin c} = \text{cotan}\, c$, ∴ this equation becomes

$$\text{cotan}\, \mathrm{C} = \sin b \times \text{cotan}\, c, \text{ or as cotan} = \frac{1}{\tan},$$

$$\frac{1}{\tan \mathrm{C}} = \frac{\sin b}{\tan c},$$

$$\therefore \tan c = \tan \mathrm{C} \times \sin b,$$

a result perfectly symmetrical with the fourth new formula, and which might have been deduced at once from that by changing B, c, b, into C, b, c, respectively.

Obs. 1. Let us collect these ten formulæ in a *body*, for easy reference, arranging them in the simplest form, thus;

1. $\sin b = \sin \mathrm{B} \times \sin a$.
2. $\sin c = \sin \mathrm{C} \times \sin a$.
3. $\tan b = \cos \mathrm{C} \times \tan a$.
4. $\tan c = \cos \mathrm{B} \times \tan a$.
5. $\cos a = \cos b \times \cos c$.
6. $\cos \mathrm{C} = \cos c \times \sin \mathrm{B}$.
7. $\cos \mathrm{B} = \cos b \times \sin \mathrm{C}$.
8. $\text{cotan}\, \mathrm{C} = \tan \mathrm{B} \times \cos a$.
9. $\tan b = \tan \mathrm{B} \times \sin c$.
10. $\tan c = \tan \mathrm{C} \times \sin b$.

The above formulæ are, each and all, equations between *three* of the five parts B, C, a, b, c, which, with the right angle A, make up a right-angled spherical triangle. Hence, any two of those parts being given, the third may be found by one or other of these equations, and so the whole triangle be determined.

Obs. 2. It would be rather difficult to retain the above ten formulæ in the memory, so as to be able to apply them whenever it was necessary. Recollection, however, would be much facilitated if we could reduce the number of equations, so as yet to include them all. That this is possible becomes immediately evident from the way in which we derived these equations, or from a bare inspection of them. For, we see that 1, 3, 6, 9, are exactly similar to 2, 4, 7, 10; and therefore we have only to recollect the former, changing B, C, b, c, respectively into C, B, c, b, when we would recollect the latter. By this means we are enabled at once to reduce the number of equations to six, instead of ten:

$$\begin{array}{rl}
\text{I.} & \sin b = \sin B \times \sin a. \\
\text{II.} & \tan b = \cos C \times \tan a. \\
\text{III.} & \cos a = \cos b \times \cos c. \\
\text{IV.} & \cos C = \cos c \times \sin B. \\
\text{V.} & \operatorname{cotan} C = \tan B \times \cos a. \\
\text{VI.} & \tan b = \tan B \times \sin c.
\end{array}$$

Such a table might be recollected without much difficulty. The omitted equations might likewise be easily supplied if necessary in practice, as is evident. For example, if sin C and sin a were given to determine c, then, in No. I, changing B into C, and b into c, we get

$$\sin c = \sin C \times \sin a,$$

the omitted equation (2), which answers the question.

But the number of these equations may be still farther reduced by a little consideration and ingenuity. We perceive that in the equations I, IV, and III, only the sines and cosines appear, while in II, VI, and V, tangents and cotangents appear also. Thus, we get the two remarkably distinct sets of formulæ:

$$\begin{array}{l|l}
\sin b = \sin B \times \sin a. & \tan b = \cos C \times \tan a. \\
\cos C = \cos c \times \sin B. & \tan b = \tan B \times \sin c. \\
\cos a = \cos b \times \cos c. & \operatorname{cotan} C = \tan B \times \cos a.
\end{array}$$

Now, if B', C', a', be respectively the complements of B, C, a, then, as $\sin B = \cos B'$, $\cos C = \sin C'$, $\cos a = \sin a'$, the first set of formulæ will become, by these changes,

$$\begin{aligned}
\sin b &= \cos B' \times \cos a', \\
\sin C' &= \cos c \times \cos B, \qquad \text{(A)} \\
\sin a' &= \cos b \times \cos c;
\end{aligned}$$

remarkable for all the left-hand quantities being *sines*, and all the right hand *cosines*.

Again: From the second set of formulæ we get

$$\cos \mathrm{C} = \frac{\tan b}{\tan a},$$
$$\sin c = \frac{\tan b}{\tan \mathrm{B}},$$
$$\cos a = \frac{\mathrm{cotan}\ \mathrm{C}}{\tan \mathrm{B}};$$

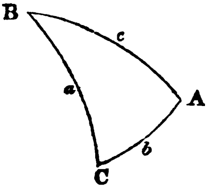
or, as $\tan = \frac{1}{\mathrm{cotan}}$, these become

$$\begin{aligned} \cos \mathrm{C} &= \tan.b \times \mathrm{cotan}\ a, \\ \sin c &= \tan b \times \mathrm{cotan}\ \mathrm{B}, \\ \cos a &= \mathrm{cotan}\ \mathrm{C} \times \mathrm{cotan}\ \mathrm{B}. \end{aligned}$$

But, cotan a = tan a', cotan B = tan B$'$, and cotan C = tan C$'$; therefore the second set of formulæ become, after these substitutions, and those above introduced,

$$\begin{aligned} \sin \mathrm{C}' &= \tan b \times \tan a', \\ \sin c &= \tan b \times \tan \mathrm{B}', \quad (\mathrm{B}) \\ \sin a' &= \tan \mathrm{C}' \times \tan \mathrm{B}'; \end{aligned}$$

which are also remarkable for all the left-hand quantities being *sines*, and all the right-hand *tangents*.



If we compare these two sets of formulæ (A) and (B), with the figure, we shall find another curious distinction between them. Of the five parts, which, with the right angle A, make up the spherical triangle, the parts b and a are immediately next to C, the parts b and B immediately next to c, the parts C and B immediately next to a; and therefore the parts C, c, a, might be called *middle* parts, while b and a, b and B, C, and B, might be called adjacent *outer*, or *extreme* parts, with respect to each of the middle ones. But, instead of this, for greater convenience, we consider C$'$, c, a', as the middle parts, while b and a', b and B$'$, C$'$ and B$'$, are considered as the extreme parts with respect to each of *these* middle ones; although, in strictness, the quantities C', a' B', are not *parts* of the triangle, being only *complements* of the true parts C, a, B. Having thus denominated these quantities, we may perceive that all the formulæ in (B) are included under one enunciation, viz.:

In any right-angled spherical triangle the sine of the middle is equal to the product of the tangents of the adjacent extremes.

For, taking C' as middle, b and a' are the adjacent extremes, and, by the formulæ (B),

$$\sin C' = \tan b \times \tan a'.$$

Taking c as middle, b and B' are the adjacent extremes, and

$$\sin c = \tan b \times \tan B'.$$

Taking a' as middle, C' and B' are the adjacent extremes, and

$$\sin a' = \tan C' \times \tan B'.$$

Likewise supplying the omitted equations, 4, 10,

$$\tan c = \cos B \times \tan a,$$
$$\tan c = \tan C \times \sin b;$$

or,

$$\cos B = \frac{\tan c}{\tan a},$$

$$\sin b = \frac{\tan c}{\tan C};$$

or,

$$\sin B' = \tan c \times \tan a',$$
$$\sin b = \tan c \times \tan C'.$$

Hence we find that going round the whole triangle, and taking b, C', a', B', c (that is, the sides about the right angle and the *complements* of the three remaining parts), each severally as *middle*, the sine of this is equal to the product of the tangents of those two which are immediately next it taken as adjacent *extremes*.

Again: we shall find that in going round the whole triangle, and taking the above parts, each severally as middle, the sine of this is equal to the product of the cosines of those two parts which are next *but one* to it, on each side. For, calling these two parts *opposite* extremes, to distinguish them from those which are immediately next to the middle part, and are therefore called *adjacent* extremes,

If we take b as middle, B' and a' are the two other parts next but one, to b, i. e. *opposite* extremes, and by (A)

$$\sin b = \cos B' \times \cos a'.$$

If we take C' as middle, c and B' are the opposite extremes, and

$$\sin C' = \cos c \times \cos B'.$$

If we take a' as middle, b and c are the opposite extremes, and

$$\sin a' = \cos b \times \cos c.$$

Likewise, supplying the omitted equations, 2, 7,

$$\sin c = \sin C \times \sin a,$$
$$\cos B = \cos b \times \sin C;$$

we have

$$\sin c = \cos C' \times \cos a',$$
$$\sin B' = \cos b \times \cos C'.$$

Which two latter formulæ show that if we take c and B' respectively as middle, in these cases also the sine of each is equal to the product of the cosines of their opposite extremes, C' and a', b and C', respectively. Hence, we may perceive that all the formulæ in (A), together with 2 and 7 (which are not absolutely specified, but implied by these formulæ), may be included under one enunciation, viz.:

In any right-angled spherical triangle, the sine of the middle is equal to the product of the cosines of the opposite extremes.

Obs. 1. From the circumstance of taking each part in order as we go *round* the triangle, these parts are denominated *circular parts.* Although the given right angle, A, is a part of the triangle, it is not considered as one of the *circular* parts; nor indeed does it enter computation at all, inasmuch as its value is known, viz.: 90°.

The above two enunciations are called *Napier's Rules*, having been first discovered by the celebrated Napier, Baron Mercheston, the inventor of Logarithms. It is not known by what process he arrived at the very remarkable truths contained in those rules; we should be almost led to suspect that it was by a happy combination of accident and conjecture. However this may be, they constitute a most useful instrument of computation.

Obs. 2. Recollecting the two preceding Rules, the reader has all that is necessary to understand the solution of right-angled spherical triangles, or the computations performed by their means in Astronomy, and other Sciences. It would be of no utility to introduce here examples of such solutions or computations, which, unlike those of Plane Trigonometry, are seldom requisite in common practice. For our readers a knowledge of the principle on which these Rules are founded is sufficient; to understand their application, and its validity, being alone requisite towards proceeding with complete satisfaction.

There are formulæ for the solution and determination of spherical triangles, whether right-angled or not; but they would be so difficult to investigate, so manifold and complex when investigated, and withal of such very rare occurrence or use in the introductory Sciences, that they may be omitted without disadvantage. We shall have no necessity to consider any spherical triangles but those which are right-angled; and for the solution of these we have provided two Rules, easy to recollect, and universally applicable.

NOTES.

NOTE A. THE sexagesimal division of the circle is that chiefly used in British works of science heretofore published; another division, which we propose to explain, has latterly been in some instances preferred, both on the continent and in England. Instead of 360, the goniometric circle is supposed divided into 400 equal parts, likewise called degrees. Each degree is divided into 100 minutes, each minute into 100 seconds, and so on. This division of the circle is called the *decimal* division, because it is carried on by certain powers of 10, as it is evident. Amongst its advantages over the sexagesimal division this is one of the least, *viz.*: an arch or angle may be expressed in degrees and decimal parts of a degree, as well as in degrees, minutes, seconds, &c. Thus,

$$24^\circ\, 35'\, 22'' = 24^\circ \cdot 3522;$$

because $24^\circ \cdot 3522^\circ = 24^\circ + \frac{3522^\circ}{10000} = 24^\circ + \frac{35^\circ}{100} + \frac{22^\circ}{10000}$; but as $1' = \frac{1^\circ}{100}$, and $1'' = \frac{1^\circ}{10000}$, $\therefore \frac{35^\circ}{100} = 35'$, and $\frac{22^\circ}{10000} = 22''$.

In our Treatise the sexagesimal division is used, not that we think it superior, but because it is that with which the English reader will meet almost universally in the course of his scientific studies; and also because there is no difficulty in transmuting degrees, minutes, &c., of one kind into those of the other, by the help of the following table:

$$1^\circ = 54',$$
$$1' = 32'' \cdot 4,$$
$$1'' = 0'' \cdot 324.$$

NOTE B. The complement of an angle was at first understood to be what the angle *wanted* to make it up 90°, and the supplement what the angle *wanted* to make it up 180°; but the definitions given in the text allow these terms an extension which is now universally attributed to them. Thus, the angle BCD′ is the complement of ACD′, because it is the difference between ACD′ and 90°, although ACD′ is greater than 90°.

Again, if we choose to consider the angular space round the point C, measured, according to the letters ABA'L, as an angle, ACL, greater than 180°, we shall have A'CL, the supplement of that angle ACL, because it is the difference between ACL and 180°. If we suppose the radius CL to revolve round C still farther, as to D again, the angular space measured according to the letters ABLD, may be considered as an angle greater than 360°; and there is, in fact, no limit to such angles made by the revolution of the radius CA round the point C. Angles of this kind are of frequent occurrence in mathematical works, but they need not enter into our calculations.

NOTE C. The truth of this latter assertion is sufficiently obvious, without a demonstration of it.

END OF THE ELEMENTS OF SPHERICAL TRIGONOMETRY.

ERRATA.

P. 3, line 4 from bottom, *for* "this series," *read* "the series which."
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